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2006 J. Phys. A: Math. Gen. 39 7825

(http://iopscience.iop.org/0305-4470/39/25/S04)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 7825–7865 doi:10.1088/0305-4470/39/25/S04

Renormalization group, operator product expansion and anomalous scaling in models of turbulent advection

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Received 29 November 2005
Published 7 June 2006
Online at stacks.iop.org/JPhysA/39/7825

Abstract

Recent progress on the anomalous scaling in models of turbulent heat and mass transport is reviewed with the emphasis on the approach based on the field-theoretic renormalization group (RG) and operator product expansion (OPE). In that approach, the anomalous scaling is established as a consequence of the existence in the corresponding field-theoretic models of an infinite number of 'dangerous' composite fields (operators) with negative critical dimensions, which are identified with the anomalous exponents. This allows one to calculate the exponents in a systematic perturbation expansion, similar to the ε expansion in the theory of critical phenomena. The RG and OPE approach is presented in a self-contained way for the example of a passive scalar field (temperature, concentration of an impurity, etc) advected by a self-similar Gaussian velocity ensemble with vanishing correlation time, the so-called Kraichnan's rapid-change model, where the anomalous exponents are known up to order $O(\varepsilon^3)$. Effects of anisotropy, compressibility and the correlation time of the velocity field are discussed. Passive advection by non-Gaussian velocity field governed by the stochastic Navier–Stokes equation and passively advected vector (e.g. magnetic) fields are considered.

PACS numbers: 47.10.+g, 47.27.−i, 05.10.Cc

1. Introduction

It has become a commonplace to emphasize that theoretical understanding of turbulence remains the last unsolved problem of classical physics. Of course, the concept of turbulence refers to a great deal of disparate physical situations ('almost as varied as in the realm of life' ([1], p 1)), and any exhaustive and ultimate 'theory of turbulence', of course, can hardly ever be established. There is, however, a classical 'list' of phenomena (or, rather, classes of phenomena) that represent and illustrate the main features of turbulence: existence and stability of solutions of hydrodynamics equations, convective turbulence, (in)stability of laminar flows and origin of turbulence, and so on. Those topics, which are of great practical and conceptual importance, have always remained in the focus of attention for theoreticians. One of them is the fully developed (homogeneous, isotropic, inertial-range) hydrodynamical turbulence. Detailed description of this concept and the bibliography of this old but still open subject can be found in the classical monographs [1–3].

Turbulent flows that occur in various liquids or gases at very high Reynolds numbers reveal a number of general aspects (cascades of energy or other conserved quantities, scaling behaviour with apparently universal 'anomalous exponents', intermittency, statistical conservation laws and so on), which support the hopes that those phenomena can be explained within a self-contained and internally consistent theory. Recent developments in this area are presented and summarized in [4].

The issue of interest is, in particular, the behaviour of the equal-time structure functions

$$S_n(r) \equiv \langle[\theta(\mathbf{x}) - \theta(\mathbf{x}')]^n\rangle, \qquad r \equiv |\mathbf{x} - \mathbf{x}'|, \tag{1.1}$$

in the inertial range of scales: $\ell \ll r \ll L$, where ℓ is the Kolmogorov dissipation length and L is the integral (external) turbulence scale L. The field $\theta(x) \equiv \theta(t, \mathbf{x})$ can be the component of the velocity field directed along the vector $\mathbf{x} - \mathbf{x}'$, or some scalar field in the problem of turbulent advection: temperature or entropy of the fluid, concentration of an impurity, etc. The angle brackets $\langle\cdots\rangle$ denote the ensemble averaging, and the time argument common to all the quantities in (1.1) is omitted.

According to the celebrated Kolmogorov–Obukhov (KO) phenomenological theory [1–3], the structure functions in the inertial range are independent of both the external and internal scales (the first and the second Kolmogorov hypothesis, respectively) and are determined by the only parameter $\bar{\epsilon}$, the mean energy dissipation rate. Dimensionality considerations then determine functions (1.1), apart from numerical coefficients, in the form

$$S_n(r) = c_n(\bar{\epsilon}r)^{n/3}. \tag{1.2}$$

The most remarkable features of developed turbulence are encoded in the single term of intermittency. This concept has no rigorous definition within the classical probabilistic theory; an excellent introduction can be found in [5] and chapter 8 of the book [1]. Roughly speaking, intermittency means that statistical properties (for example, correlation or structure functions of the turbulent velocity field) are dominated by rare spatiotemporal configurations, in which the regions with strong turbulent activity have exotic (fractal) geometry and are embedded into the vast regions with regular (laminar) flow.

In the turbulence, such a phenomenon is believed to be related to strong fluctuations of the energy flux. Therefore, it leads to deviations from the predictions of the KO theory. Such deviations, referred to as 'anomalous' or 'non-dimensional' scaling, manifest themselves in singular (arguably power-like) dependence of correlation functions on the distances and the integral scale L (in contradiction with the first Kolmogorov hypothesis). For functions (1.1), they are written in the form

$$S_n(r) = c_n(\bar{\epsilon}r)^{n/3}(r/L)^{q_n}. \tag{1.3}$$

The 'anomalous exponents' q_n are certain nontrivial and nonlinear functions of n, the order of the structure function: the phenomenon also referred to as 'multiscaling'.

Within the framework of numerous semi-heuristic models, the anomalous exponents are related to statistical properties of the local dissipation rate, the fractal (Haussdorf) dimension of structures formed by the small-scale turbulent eddies, the characteristics of nontrivial structures (vortex filaments) and so on; see [1–3] for a review and further references. The common drawback of such models is that they are only loosely related to underlying hydrodynamical equations, involve arbitrary adjusting parameters and, therefore, cannot be considered to be the basis for construction of a systematic perturbation theory in certain small (at least formal) expansion parameter; see e.g. the remark in [6]. Thus, serious doubts remain about the universality of anomalous exponents and the very existence of deviations from the KO theory.

The term 'anomalous scaling' reminds of the critical scaling in models of equilibrium phase transitions. In those, the field-theoretic renormalization group (RG) was successfully employed to establish the existence of self-similar (scaling) regimes and to construct regular perturbative calculational schemes (the famous ε expansion and its relatives) for the corresponding exponents, scaling functions, ratios of amplitudes, etc; see e.g. [7, 8] and references therein.

In fact, the analogy is far from exact. There is an important difference between the concepts of critical scaling in equilibrium phase transitions and anomalous scaling in turbulence. Formally speaking, in both the cases one deals with nontrivial powers of the distance, but in the first case they are divided by the ultraviolet (UV) scale ℓ, while in the second the same role is played by the integral or infrared (IR) scale L. The aforementioned phenomenon of multiscaling was also often opposed to critical scaling, because in the latter 'everything is determined by just two exponents η and ν'.

It was hoped that a close analogy can be achieved if the momentum space for turbulence be confronted with the coordinate space for critical phenomena. This idea was expressed in a phenomenological 'dictionary', where, in particular, the viscous length ℓ (that is, the UV scale of turbulence) was confronted with the correlation length (that is, the IR scale of critical phenomena), while the integral scale L was confronted with the molecular length (see e.g. [9]); hence the idea of 'inverse' renormalization group (see [10, 11] for a recent discussion).

Both the natural and numerical experiments suggest that the deviation from the classical KO theory is even more strongly pronounced for passively advected scalar fields than for the velocity field itself; see e.g. [12–19] and literature cited therein. At the same time, the problem of passive advection appears easier tractable theoretically: even simplified models describing the advection by a 'synthetic' velocity field with a given Gaussian statistics reproduce many of the anomalous features of genuine turbulent heat or mass transport observed in experiments. Therefore, the problem of passive scalar advection, being of practical importance in itself, may also be viewed as a starting point in studying intermittency and anomalous scaling in the turbulence as a whole.

Probably, the most important progress in the subject, achieved in the last decade of the 20th century, was related to a simplified model of the fully developed turbulence, known as Kraichnan's rapid-change model. The model, which dates back to classical studies of Batchelor, Obukhov, Kraichnan and Kazantzev, describes a scalar quantity passively advected by a Gaussian velocity field, decorrelated in time and self-similar in space (the latter property mimics some features of a real turbulent velocity ensemble). The pair correlation function of the velocity is taken in the form $\langle v_i v_j \rangle \propto D_0 \delta(t - t') k^{-d-\varepsilon}$, where k is the wave number, d is the space dimensionality and ε is an arbitrary parameter. For the first time, the existence of anomalous scaling was established on the basis of a microscopic model [20], and the corresponding anomalous exponents were derived within controlled approximations [21, 22] and a regular perturbation scheme [23]. Namely, it was shown that the inertial-range structure functions in such a model exhibit anomalous scaling behaviour:

$$S_{2n}(r) \propto D_0^{-n} r^{n(2-\varepsilon)} (r/L)^{\Delta_n} \tag{1.4}$$

(the odd functions vanish) with negative anomalous exponents Δ_n, whose first terms of the expansion in ε [21] and $1/d$ [22] have the forms

$$\Delta_n = -2n(n-1)\varepsilon/(d+2) + O(\varepsilon^2) = -2n(n-1)\varepsilon/d + O(1/d^2). \qquad (1.5)$$

The relation $\Delta_1 = 0$ is exact, in agreement with the exact solution for the pair correlator derived in [24].

Another quantity of interest is the local dissipation rate of scalar fluctuations, $E(x) = \kappa_0 \partial_i \theta(x) \partial_i \theta(x)$, where κ_0 is the diffusivity coefficient. The equal-time correlation functions of its powers in the inertial range have the forms [21, 22]

$$\langle E^n(x) E^p(x') \rangle \propto (r/\ell)^{-\Delta_n - \Delta_p} (mr)^{\Delta_{n+p}} \qquad (1.6)$$

with the same exponents Δ_n from (1.5). Relations of the form (1.6) are characteristic of the models with multifractal behaviour [25, 26].

Fortunately, that there are (at least) two alternative (or complementary) analytical approaches to the rapid-change model. The 'zero-mode approach', developed in [21, 22], can be interpreted as a realization of the well-known idea of self-consistent (bootstrap) equations, which involve skeleton diagrams with dressed lines and dropped bare terms. Owing to special features of the rapid-change models (linearity in the passive field and time decorrelation of the advecting field), such equations can be written in a closed form, as certain differential equations for the equal-time correlation functions. In this sense, the model is 'exactly solvable'. Although those equations cannot be solved explicitly, the nontrivial anomalous exponents (1.5) can be extracted from the analysis of the asymptotic behaviour for $(r/L) \ll 1$ of their zero modes (unforced solutions) in the limit $\varepsilon \to 0$ [21] or $1/d \to 0$ [22].

From a more physical point of view, zero modes can be interpreted as statistical conservation laws in the dynamics of particle clusters [27]. The concept of statistical conservation laws appears rather general, being also confirmed by numerical simulations of [28, 29], where the passive advection in the two-dimensional Navier–Stokes (NS) velocity field [28] and a shell model of a passive scalar [29] were studied. This observation is rather intriguing because in those models no closed equations for equal-time quantities can be derived due to the fact that the advecting velocity has a finite correlation time.

The second systematic analytical approach to the rapid-change model, proposed in [23], is based on the field-theoretic renormalization group (RG) and operator product expansion (OPE).

To avoid possible confusion, it should be explained that in [23] and subsequent papers, the conventional renormalization group (and not the inverse RG in the spirit of [9–11]) was employed, which is based on the standard renormalization procedure (elimination of UV divergences). The solution proceeds in two main stages. In the first stage, the multiplicative renormalizability of the corresponding field-theoretic model is demonstrated and the differential RG equations for its correlation functions are obtained. The asymptotic behaviour of the latter on their UV argument (r/ℓ) for $r \gg \ell$ and any fixed (r/L) is given by IR-stable fixed points of those equations. It involves some 'scaling functions' of the IR argument (r/L), whose form is not determined by the RG equations. In the second stage, their behaviour at $r \ll L$ is found from the OPE within the framework of the general solution of the RG equations. There, the crucial role is played by the critical dimensions of various composite fields (composite operators in quantum-field terminology) which give rise to an infinite family of independent scaling exponents—and hence to multiscaling.

Of course, both these stages have long been known in the RG theory of critical behaviour, where the OPE is used in the analysis of the small-(r/L) form of the scaling functions; see e.g. [7, 8] and references therein. The distinguishing feature, specific to models of turbulence, is the existence of composite operators with *negative* critical dimensions. Such operators sometimes are termed 'dangerous', because their contributions to the OPE diverge at $(r/L) \to 0$.

In the models of critical phenomena, nontrivial composite operators always have strictly positive dimensions, so that they only determine corrections (vanishing for $(r/L) \to 0$) to the leading terms (finite for $(r/L) \to 0$) in the scaling functions (the leading terms are related to the simplest operator unity with zero critical dimension).

The OPE and the concept of dangerous operators in the stochastic hydrodynamics were introduced and investigated in detail in [30]; detailed discussion of the NS case can be found in the review paper [31], the monograph [32] and chapter 6 of the book [8]. Later, the idea of negative dimensions was repeatedly introduced in connection with the anomalous scaling in turbulence [26], models with multifractal behaviour [25] and the phenomena related to the Burgers equation [33, 34].

The RG analysis of [23] has shown that dangerous operators are indeed present in the rapid-change model, and that their dimensions can be calculated systematically within a regular perturbation expansion, similar to the famous ε expansion of the critical exponents. Owing to the linearity of the original stochastic equations in the passive field, only finite number of dangerous operators can contribute to any given structure function, which allows one to identify the corresponding anomalous exponent with the critical dimension of an individual composite operator. The actual calculations were performed to the second [23] and third [35, 36] orders in ε (two-loop and three-loop approximations, respectively). Generalizations to the cases of compressible [37, 38] and anisotropic [39, 40] velocity ensembles and the vector advected field [41–45] have been obtained.

The two approaches complement each other nicely: the zero-mode technique allows for exact (nonperturbative) solutions for the anomalous exponents related to second-order correlation functions [22, 47–49] (they are nontrivial for passive vector fields or anisotropic sectors for scalar fields), while the RG approach forms the basis for systematic perturbative calculations of the higher order anomalous exponents [23, 35–38]. For the cases of anisotropic velocity ensembles or/and passively advected vector fields, as well as advection of extended objects, where the calculations become rather involved, all the existing results for higher order correlation functions were derived only by means of the RG approach and only to the leading order in ε [39, 41–46].

Detailed discussion of the zero-mode approach, the concept of statistical conservation laws, Lagrangian description of the passive advection and detailed bibliography of the subject can be found in the review paper [19] and the lectures [50]. In the present paper, we focus on the RG and OPE approach to the problem. In section 2, we give a brief but self-contained exposition of this approach for the simplest example: passive scalar field advected by the incompressible, isotropic and homogeneous Kraichnan's velocity ensemble.

Existence of exact solutions, regular perturbation schemes and accurate numerical simulations allows one to discuss, for the example of the rapid-change model and its relatives, the issues that are interesting within the general context of fully developed turbulence: universality and saturation of anomalous exponents, effects of compressibility, anisotropy and pressure, persistence of the large-scale anisotropy and hierarchy of anisotropic contributions, convergence properties and nature of the ε expansions, and so on. These issues are discussed in section 3.

Besides the calculational efficiency, an important advantage of the RG approach is its relative universality: it is not bound to the aforementioned 'solvability' of the rapid-change model and can also be applied to the case of finite correlation time [51–54] or non-Gaussian advecting field governed by the stochastic Navier–Stokes equation [51, 55]. These issues are discussed in section 4. Turbulent advection of vector (e.g. magnetic) fields is considered in section 5. The lessons we have learned from the RG analysis of the passive advection and open problems are briefly discussed in section 6.

2. Anomalous scaling in the Obukhov–Kraichnan model

The turbulent advection of a passive scalar field $\theta(x) \equiv \theta(t, \mathbf{x})$ is described by the stochastic equation

$$\nabla_t\theta = \kappa_0\partial^2\theta + f, \qquad \nabla_t \equiv \partial_t + v_i\partial_i, \tag{2.1}$$

where $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, κ_0 is the molecular diffusivity coefficient, ∂^2 is the Laplace operator, $\mathbf{v}(x) \equiv \{v_i(x)\}$ is the transverse (owing to the incompressibility condition $\partial_i v_i = 0$) velocity field and $f \equiv f(x)$ is an artificial Gaussian scalar noise with zero mean and correlation function

$$\langle f(x)f(x')\rangle = \delta(t-t')C(r/L), \qquad r = |\mathbf{r}|, \quad \mathbf{r} = \mathbf{x}-\mathbf{x}'. \tag{2.2}$$

The parameter L is an integral scale related to the noise and $C(r/L)$ is some function finite as $(r/L) \to 0$ (with no loss of generality, we will set $C(0) = 1$).

In the real problem, the field $\mathbf{v}(x)$ satisfies the Navier–Stokes equation. In the rapid-change model, it obeys a Gaussian distribution with zero mean and correlation function

$$\langle v_i(x)v_j(x')\rangle = D_0\delta(t-t')\int_{k>m}\frac{\mathrm{d}\mathbf{k}}{(2\pi)^d}P_{ij}(\mathbf{k})\exp[\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')], \tag{2.3}$$

where $P_{ij}(\mathbf{k}) = \delta_{ij} - k_ik_j/k^2$ is the transverse projector, $k \equiv |\mathbf{k}|$, $D_0 > 0$ is an amplitude factor, d is the dimensionality of the $\mathbf{x}$ space and $0 < \varepsilon < 2$ is a parameter with the real ('Kolmogorov') value $\varepsilon = 4/3$.

The IR regularization is provided by the cut-off in the integral (2.3) from below at $k = m$, where $m \equiv 1/L$ is the reciprocal of the integral scale L (for simplicity, we do not distinguish the IR scales related to the noise and the velocity). The anomalous exponents are independent of the precise form of the IR regularization; the sharp cut-off is the most convenient choice from the calculational viewpoints (another possibility is the replacement $k^2 \to k^2 + m^2$ in the denominator of (2.3)). The relations

$$D_0/\kappa_0 = g_0 = \Lambda^\varepsilon \tag{2.4}$$

introduce the coupling constant g_0 (the formal expansion parameter in the ordinary perturbation theory) and the characteristic UV momentum scale $\Lambda \equiv 1/\ell$.

2.1. Field-theoretic formulation

The stochastic problem (2.1)–(2.3) is equivalent to the field-theoretical model of the set of three fields $\Phi \equiv \{\theta, \theta', \mathbf{v}\}$ with action functional

$$\mathcal{S}(\Phi) = \theta' D_\theta\theta'/2 + \theta'[-\partial_t\theta - (v_i\partial_i)\theta + \kappa_0\partial^2\theta] - \mathbf{v}D_v^{-1}\mathbf{v}/2. \tag{2.5}$$

The first four terms in (2.5) represent the De Dominicis–Janssen-type action [56] for the stochastic problem (2.1), (2.2) at fixed $\mathbf{v}$, while the last term corresponds to the Gaussian averaging over $\mathbf{v}$ with correlator (2.3). Here D_θ and D_v are the correlators (2.2) and (2.3), respectively, and the required integrations over $x = (t, \mathbf{x})$ and summations over the vector indices are implied.

This formulation means that statistical averages of random quantities in the original stochastic problem (2.1)–(2.3) coincide with the Green functions of the field-theoretic model with action (2.5), given by functional averages with the weight $\exp\mathcal{S}(\Phi)$.

The action (2.5) corresponds to a standard Feynman diagrammatic technique with the triple vertex $-\theta'(v_i\partial_i)\theta$ and bare propagators (in the momentum–frequency representation)

$$\begin{aligned}&\langle\theta\theta'\rangle_0 = \langle\theta'\theta\rangle_0^* = (-\mathrm{i}\omega + \kappa_0k^2)^{-1}, \qquad \langle\theta'\theta'\rangle_0 = 0,\\ &\langle\theta\theta\rangle_0 = C(k)\big(\omega^2 + \kappa_0^2k^4\big)^{-1},\end{aligned} \tag{2.6}$$

Table 1. Canonical dimensions of the fields and parameters in the model (2.5).

F	θ	θ'	$\mathbf{v}$	κ, κ_0	m, μ, Λ	g_0	g
d_F^k	0	d	-1	-2	1	ε	0
d_F^ω	$-1/2$	$1/2$	1	1	0	0	0
d_F	-1	$d+1$	1	0	1	ε	0

where $C(k)$ is the Fourier transform of the function $C(r/L)$ in (2.2) and the bare propagator $\langle \mathbf{vv} \rangle_0$ is given by equation (2.3). The role of the coupling constant in the perturbation theory is played by the parameter g_0 defined in (2.4).

2.2. UV singularities and renormalization

It is well known that the analysis of UV divergences is based on the analysis of canonical dimensions ('power counting'); see e.g. [7, 8]. Dynamical models of the type (2.5), in contrast to static models, have two scales, i.e., the canonical dimension of some quantity F (a field or a parameter in the action functional) is described by two numbers, the momentum dimension d_F^k and the frequency dimension d_F^ω. They are determined so that $[F] \sim [L]^{-d_F^k}[T]^{-d_F^\omega}$, where L is the length scale and T is the time scale. The dimensions are found from the obvious normalization conditions $d_k^k = -d_{\mathbf{x}}^k = 1$, $d_k^\omega = d_{\mathbf{x}}^\omega = 0$, $d_\omega^k = d_t^k = 0$, $d_\omega^\omega = -d_t^\omega = 1$, and from the requirement that each term of the action functional be dimensionless (with respect to the momentum and frequency dimensions separately). Then, based on d_F^k and d_F^ω, one can introduce the total canonical dimension $d_F = d_F^k + 2d_F^\omega$ (in the free theory, $\partial_t \propto \partial^2$), which plays in the theory of renormalization of dynamical models the same role as the conventional (momentum) dimension does in static problems. Renormalization of dynamic models is discussed in chapter 5 of [8] in detail.

The dimensions for the model (2.5) are given in table 1, including renormalized parameters, which will be introduced later on. From table 1, it follows that the model is logarithmic (the coupling constant g_0 is dimensionless) at $\varepsilon = 0$, so that the UV divergences in the correlation functions have the form of the poles in ε.

The total canonical dimension of an arbitrary 1-irreducible Green function $\Gamma = \langle \Phi \cdots \Phi \rangle_{\text{1-ir}}$ is given by the relation $d_\Gamma = d_\Gamma^k + 2d_\Gamma^\omega = d + 2 - N_\Phi d_\Phi$, where $N_\Phi = \{N_\theta, N_{\theta'}, N_{\mathbf{v}}\}$ are the numbers of corresponding fields entering into the function Γ, and the summation over all types of the fields is implied. The total dimension d_Γ is the formal index of the UV divergence. Superficial UV divergences, whose removal requires counterterms, can be present only in those functions Γ for which d_Γ is a non-negative integer.

Analysis of the divergences should be based on the following auxiliary considerations:

(1) From the explicit form of the vertex and bare propagators in the model (2.5), it follows that $N_{\theta'} - N_\theta = 2N_0$ for any 1-irreducible Green function, where $N_0 \geqslant 0$ is the total number of the bare propagators $\langle \theta\theta \rangle_0$ entering into the function (obviously, no diagrams with $N_0 < 0$ can be constructed). Therefore, the difference $N_{\theta'} - N_\theta$ is an even non-negative integer for any nonvanishing function.

(2) Diagrams for some Green functions contain closed circuits of retarded propagators $\langle \theta\theta' \rangle_0$ and also vanish. These are, for example, all the 1-irreducible functions with $N_{\theta'} = 0$.

(3) If for some reason a number of external momenta occur as an overall factor in all the diagrams of a given Green function, the real index of divergence d'_Γ is smaller than d_Γ by the corresponding number of unities (the Green function requires counterterms only if d'_Γ is a non-negative integer).

In the model (2.5), the derivative ∂ at the vertex $\theta'(v_i\partial_i)\theta$ can be moved onto the field θ' by virtue of the transversality of the field $\mathbf{v}$. Therefore, in any 1-irreducible diagram it is always possible to move the derivative onto any of the external tails θ or θ', which decreases the real index of divergence: $d'_\Gamma = d_\Gamma - N_\theta - N_{\theta'}$. The fields θ, θ' enter into the counterterms only in the form of derivatives $\partial\theta, \partial\theta'$.

From the dimensions in table 1, we find $d_\Gamma = d + 2 - N_{\mathbf{v}} + N_\theta - (d+1)N_{\theta'}$ and $d'_\Gamma = (d+2)(1 - N_{\theta'}) - N_{\mathbf{v}}$. From these expressions, we conclude that for any d superficial divergences can exist only in the 1-irreducible functions $\langle\theta'\theta\cdots\theta\rangle$ with $N_{\theta'} = 1$ and arbitrary value of N_θ, for which $d_\Gamma = 2, d'_\Gamma = 0$. However, all the functions with $N_\theta > N_{\theta'}$ vanish (see above) and obviously do not require counterterms. We are left with the only superficially divergent function $\langle\theta'\theta\rangle$; the corresponding counterterm must contain two symbols ∂ and is therefore reduced to $\theta'\partial^2\theta$. This leads to the renormalized action of the form

$$\mathcal{S}_R(\Phi) = \theta' D_\theta \theta'/2 + \theta'[-\partial_t\theta - (v_i\partial_i)\theta + \kappa Z_\kappa \partial^2\theta] - \mathbf{v}D_v^{-1}\mathbf{v}/2 \tag{2.7}$$

or, equivalently, to the multiplicative renormalization of the parameters κ_0 and g_0 in the action functional (2.5) with the only independent renormalization constant Z_κ:

$$\kappa_0 = \kappa Z_\kappa, \qquad g_0 = g\mu^\varepsilon Z_g, \qquad Z_g = Z_\kappa^{-1}. \tag{2.8}$$

Here μ is the reference mass in the minimal subtraction scheme (MS), which we always use in what follows, g and κ are renormalized analogues of the bare parameters g_0 and κ_0, and $Z = Z(g, \varepsilon, d)$ are the renormalization constants. Their relation in (2.8) results from the absence of renormalization of the contribution with D_0 in (2.5), so that $D_0 \equiv g_0\kappa_0 = g\mu^\varepsilon\kappa$, see (2.4). No renormalization of the fields and the 'mass' m is required, i.e., $Z_\Phi = 1$ for all Φ and $m_0 = m$, $Z_m = 1$.

2.3. RG equations, RG functions and the fixed point

The fields in our model are not renormalized, their renormalized Green functions W^R coincide with the corresponding unrenormalized functions $W = \langle\Phi\cdots\Phi\rangle$; the only difference is in the choice of variables and in the form of perturbation theory (in g instead of g_0):

$$W^R(g, \kappa, \mu, \ldots) = W(g_0, \kappa_0, \ldots) \tag{2.9}$$

(the dots stand for other arguments such as coordinates and momenta).

We use $\widetilde{\mathcal{D}}_\mu$ to denote the differential operator $\mu\partial_\mu$ for fixed bare parameters g_0, κ_0 and operate on both sides of equation (2.9) with it. This gives the basic differential RG equation

$$\mathcal{D}_{\mathrm{RG}} W^R(g, \kappa, \mu, \ldots) = 0, \qquad \mathcal{D}_{\mathrm{RG}} \equiv \mathcal{D}_\mu + \beta(g)\partial_g - \gamma_\kappa(g)\mathcal{D}_\kappa, \tag{2.10}$$

where we have written $\mathcal{D}_s \equiv s\partial_s$ for any variable s, $\mathcal{D}_{\mathrm{RG}}$ is the operation $\widetilde{\mathcal{D}}_\mu$ expressed in renormalized variables and the RG functions (the β function and the anomalous dimension γ) are defined as

$$\gamma_\kappa(g) \equiv \widetilde{\mathcal{D}}_\mu \ln Z_\kappa, \qquad \beta(g) \equiv \widetilde{\mathcal{D}}_\mu g = g[-\varepsilon + \gamma_\kappa]. \tag{2.11}$$

The relation between β and γ results from the definitions and the last relation in (2.8). In general, if some quantity F is renormalized multiplicatively, $F = Z_F F^R$, it satisfies the RG equation of the form

$$[\mathcal{D}_{\mathrm{RG}} + \gamma_F(g)]F^R = 0, \qquad \gamma_F(g) \equiv \mathcal{D}_\mu \ln Z_F \tag{2.12}$$

with the operator $\mathcal{D}_{\mathrm{RG}}$ from (2.10).

The constant Z_κ is determined from the condition that the exact response function $G \equiv \langle\theta\theta'\rangle$ be finite at $\varepsilon = 0$ when expressed in renormalized variables. The function G satisfies the standard Dyson equation

$$G^{-1}(\omega, \mathbf{k}) = -\mathrm{i}\omega + \kappa_0 k^2 - \Sigma_{\theta'\theta}(\omega, \mathbf{k}), \tag{2.13}$$

$$\Sigma_{\theta'\theta} =$$ 

Figure 1. The one-loop approximation to the self-energy operator in model (2.5).

where $\Sigma_{\theta'\theta}$ the self-energy operator represented by 1-irreducible diagrams. It is shown in figure 1 in the one-loop approximation. There, the solid line represents the bare propagator $\langle\theta\theta'\rangle_0$, slashed ends correspond to the field θ', ends without a slash correspond to θ; the dashed line represents the correlator (2.3); the points where it is attached to the solid line correspond to the vertex $-\theta'(v_i\partial_i)\theta$. From the explicit form of the vertex and the bare propagators (2.3), (2.6), it follows that any multi-loop diagram of the function $\Sigma_{\theta'\theta}$ contains effectively a closed circuit of retarded propagators $\langle\theta\theta'\rangle_0$ and therefore vanishes; it is crucial here that the propagator $\langle\mathbf{vv}\rangle_0$ in (2.3) is proportional to the δ function in time. Thus, the function $\Sigma_{\theta'\theta}$ and the renormalization constant Z_κ in the model (2.5) are given by the one-loop approximation exactly, i.e., they have no corrections of orders g^2, g^3 and so on. Explicit calculation gives

$$Z_\kappa = 1 - \frac{g(d-1)C_d}{2d\varepsilon}, \tag{2.14}$$

where $C_d \equiv S_d/(2\pi)^d$ and $S_d \equiv 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the unit sphere in the d-dimensional space.

From definitions (2.11), using equation (2.14) we find exact expressions for the basic RG functions:

$$\gamma_\kappa(g) = \frac{g(d-1)C_d}{2d}, \qquad \beta(g) = g\left[-\varepsilon + \frac{g(d-1)C_d}{2d}\right]. \tag{2.15}$$

From (2.15), it follows that an IR-attractive fixed point

$$g_* = \frac{2d\varepsilon}{C_d(d-1)} \tag{2.16}$$

of the RG equations ($\beta(g_*) = 0$, $\beta'(g_*) = \varepsilon > 0$) exists in the physical region $g > 0$ for all $0 < \varepsilon < 2$. The value of $\gamma_\kappa(g)$ at the fixed point is also found exactly:

$$\gamma_\kappa^* \equiv \gamma_\kappa(g_*) = \varepsilon, \tag{2.17}$$

without corrections of orders ε^2, ε^3 and so on.

It should be noted that vanishing of the higher order terms in $\beta(g)$ and g_* is not crucial for the applicability of the RG approach: for finite correlation time [51–54] or non-Gaussian velocity ensemble [51, 55], those quantities are given by infinite series in g and ε, respectively, but the analogue of equation (2.17) holds as a consequence of the relation between the renormalization constants in (2.8).

2.4. Solution of the RG equations: invariant variables

Consider the solution of the RG equation on the example of the even different-time structure functions

$$S_{2n}(r,\tau) \equiv \langle[\theta(t,\mathbf{x}) - \theta(t',\mathbf{x}')]^{2n}\rangle, \qquad r = |\mathbf{x}-\mathbf{x}'|, \quad \tau \equiv t - t'. \tag{2.18}$$

It satisfies the RG equation $\mathcal{D}_{\mathrm{RG}}S_{2n} = 0$ with the operator $\mathcal{D}_{\mathrm{RG}}$ from equation (2.10). It should be noted that the structure functions (2.18) involve composite operators θ^n, and the above RG equation does not follow automatically from equation (2.10) for ordinary correlation functions; it will be justified later in section 2.5.

In renormalized variables, dimensionality considerations give

$$S_{2n}(\mathbf{r},\tau) = \kappa^{-n} r^{2n} R_{2n}(\mu r, \tau\kappa/r^2, mr, g), \tag{2.19}$$

where R_{2n} is a function of completely dimensionless arguments (the dependence on d and ε is implied). From the RG equation, the identical representation follows,

$$S_{2n}(\mathbf{r},\tau) = (\bar{\kappa})^{-n} r^{2n} R_{2n}(1, \tau\bar{\kappa}/r^2, mr, \bar{g},), \tag{2.20}$$

where the invariant variables $\bar{e} = \bar{e}(\mu r, e)$ satisfy the equation $\mathcal{D}_{\mathrm{RG}}\bar{e} = 0$ and the normalization conditions $\bar{e} = e$ at $\mu r = 1$ (here $e \equiv \{\kappa, g, m\}$ denotes the full set of renormalized parameters). The identity $\bar{m} \equiv m$ is a consequence of the absence of $\mathcal{D}_m$ in the operator $\mathcal{D}_{\mathrm{RG}}$ owing to the fact that m is not renormalized. Equation (2.20) is valid because both sides of it satisfy the RG equation and coincide for $\mu r = 1$ owing to the normalization of the invariant variables. The relation between the bare and invariant variables has the form

$$\kappa_0 = \bar{\kappa} Z_\kappa(\bar{g}), \qquad g_0 = \bar{g} r^{-\varepsilon} Z_g(\bar{g}). \tag{2.21}$$

Equation (2.21) determines implicitly the invariant variables as functions of the bare parameters; it is valid because both sides of it satisfy the RG equation and because equation (2.21) at $\mu r = 1$ coincides with (2.8) owing to the normalization conditions.

It is well known that for $\mu r \to \infty$ the invariant coupling constant approaches the IR-attractive fixed point: $\bar{g} \to g_*$. Furthermore, the large-μr behaviour of the invariant diffusivity $\bar{\kappa}$ is also found explicitly from equation (2.21) and the last relation in (2.8): $\bar{\kappa} = D_0 r^\varepsilon/\bar{g} \to D_0 r^\varepsilon/g_*$ (we recall that $D_0 = g_0\kappa_0$). Then for $\mu r \to \infty$ and any fixed mr, we obtain

$$S_{2n}(\mathbf{r},\tau) = (D_0/g_*)^{-n} r^{n(2-\varepsilon)} \xi_{2n}(\tau D_0 r^{\Delta_t}, mr), \tag{2.22}$$

where

$$\xi_{2n}(D_0\tau r^{\Delta_t}, mr) \equiv R_{2n}(1, D_0\tau r^{\Delta_t}, mr, g_*) \tag{2.23}$$

and $\Delta_t \equiv -2 + \gamma_\kappa^* = -2 + \varepsilon$ is the critical dimension of time. The dependence of the scaling function ξ_{2n} on its arguments is *not* determined by the RG equation (2.10) itself. For the equal-time structure function (1.1), the first argument of ξ_{2n} in the representation (2.23) is absent:

$$S_{2n}(\mathbf{r}) = (D_0/g_*)^{-n} r^{n(2-\varepsilon)} \xi_{2n}(mr), \tag{2.24}$$

where the definition of ξ_{2n} is obvious from (2.23). It is noteworthy that equations (2.22)–(2.24) prove the independence of the structure functions in the IR range (large μr and arbitrary mr) of the diffusivity coefficient or, equivalently, of the UV scale: the parameters g_0 and κ_0 enter into equation (2.22) only in the form of the combination $D_0 = g_0\kappa_0$. A similar property was established in [57] for the stirred Navier–Stokes equation and is related to the second Kolmogorov hypothesis; see also the discussion in [30–32].

Now let us turn to the general case. Let $F(r,\tau)$ be some multiplicatively renormalized quantity (say, a correlation function involving composite operators), i.e., $F = Z_F F_R$ with certain renormalization constant Z_F. It satisfies the RG equation of the form $[\mathcal{D}_{\mathrm{RG}} + \gamma_F]F_R = 0$ with γ_F from (2.12). Dimensionality considerations give

$$F_R(r,\tau) = \kappa^{d_F^\omega} r^{-d_F} R_F(\mu r, \tau\kappa/r^2, mr, g), \tag{2.25}$$

where d_F^ω and d_F are the frequency and total canonical dimensions of F (see section 2.2) and R_F is a function of dimensionless arguments. The analogue of equation (2.20) has the form

$$F(r,\tau) = Z_F(g)F_R = Z_F(\bar{g})(\bar{\kappa})^{d_F^\omega} r^{-d_F} R_F(1, \tau\bar{\kappa}/r^2, mr, \bar{g}). \tag{2.26}$$

In the large-μr limit, one has $Z_F(\bar{g}) \simeq \mathrm{const}(\Lambda r)^{-\gamma_F^*}$. The UV scale appears in this relation from equation (2.4). Then in the IR range ($\Lambda r \sim \mu r$ large, mr arbitrary), equation (2.26) takes on the form

$$F(r,\tau) \simeq \mathrm{const}\, \Lambda^{-\gamma_F^*} D_0^{d_F^{\omega}} r^{-\Delta[F]} \xi_F(D_0 \tau r^{\Delta_t}, mr). \tag{2.27}$$

Here

$$\Delta[F] \equiv \Delta_F = d_F^k - \Delta_t d_F^{\omega} + \gamma_F^*, \qquad \Delta_t = -2 + \varepsilon, \tag{2.28}$$

is the critical dimension of the function F and the scaling function ξ_F is related to R_F as in equation (2.22). For nontrivial γ_F^*, the function F in the IR range retains the dependence on $\Lambda = 1/\ell$ or, equivalently, on κ_0.

Representation (2.24) for any scaling function $\xi(u)$ describes the behaviour of the structure functions for $\mu r \gg 1$ and any fixed value of $u \equiv mr$; the inertial range corresponds to the additional condition $u \ll 1$. As already mentioned, the form of the functions $\xi(u)$ is not determined by the RG equation (2.10). Calculating the function R in (2.23) within the renormalized perturbation theory, $R = \sum_{n=0}^{\infty} g^n R_n$, substituting $g \to g_*$ and expanding g_* and R_n in ε, one obtains the ε expansion for the scaling function (it is important here that the coefficients R_n have no poles in ε):

$$\xi(u) = \sum_{k=0}^{\infty} \varepsilon^k \xi^{(k)}(u). \tag{2.29}$$

Although the coefficients $\xi^{(k)}$ can be finite at $u = 0$, this does not prove the finiteness of $\xi(u)$ beyond the ε expansion: one can show that for any arbitrarily small value of ε there are diagrams that diverge at $m \propto u \to 0$. As a result, the coefficients $\xi^{(k)}$ contain IR singularities of the form $u^p \ln^q u$, these 'large IR logarithms' compensate for the smallness of ε, and the actual expansion parameter appears to be $\varepsilon \ln u$ rather than ε itself. Thus, the plain expansion (2.29) is not suitable for the analysis of the small-u behaviour of $\xi(u)$.

The formal statement of the problem is to sum up the expansion (2.29) assuming that ε is small with the additional condition that $\varepsilon \ln u \sim 1$. By analogy with the theory of critical behaviour [7, 8], this problem can be attacked with the aid of the operator product expansion; see [30–32]. It will be discussed in section 2.6. The key role will be played by the scaling dimensions of certain composite fields. The renormalization of those objects is discussed in the next section.

2.5. Composite fields: renormalization and scaling dimensions

In the following, an important role will be played by the composite fields (composite operators in quantum-field terminology) built of the field $\theta(x)$ and its spatial derivatives. We recall that the term 'local composite operator' refers to any monomial or polynomial built of the fields Φ and their derivatives at a single spacetime point $x = \{t, \mathbf{x}\}$, for example $\theta^n(x)$, $\theta'\partial^2\theta(x)$ or $\theta'(v\partial)\theta(x)$.

Coincidence of the field arguments in correlation functions containing an operator F gives rise to additional UV divergences, removed by a special renormalization procedure. Owing to the renormalization, the critical dimension Δ_F associated with certain operator F is not in general equal to the simple sum of critical dimensions of the fields and derivatives entering into F. As a rule, composite operators 'mix' in renormalization, that is, an UV finite renormalized operator is a linear combination of unrenormalized operators, and vice versa.

In general, counterterms to a given operator F are determined by all possible 1-irreducible Green functions with one operator F and arbitrary number of primary fields,

$\Gamma = \langle F(x)\Phi(x_1)\cdots\Phi(x_n)\rangle_{\text{1-ir}}$. The total canonical dimension (formal index of divergence) for such quantities is given by

$$d_\Gamma = d_F - \sum_\Phi N_\Phi d_\Phi, \tag{2.30}$$

with the summation over all types of fields entering into the function and the canonical dimensions from table 1. For superficially divergent diagrams, d_Γ is a non-negative integer.

Consider first operators of the form $\theta^n(x)$ with the canonical dimension $d_F = -n$, entering the structure functions (1.1). From table 1, we obtain $d_\Gamma = -n + N_\theta - N_{\mathbf{v}} - (d+1)N_{\theta'}$, and from the analysis of the diagrams it follows that the total number of the fields θ entering the function Γ can never exceed the number of the fields θ in the operator θ^n itself: $N_\theta \leqslant n$. Therefore, the divergence can only exist in the functions with $N_{\mathbf{v}} = N_{\theta'} = 0$ and arbitrary value of $n = N_\theta$, for which the formal index vanishes: $d_\Gamma = 0$. However, at least one of N_θ external 'tails' of the field θ is attached to a vertex $\theta'(v_i\partial_i)\theta$ (it is impossible to construct nontrivial superficially divergent diagram of the desired type with all the external tails attached to the vertex F), at least one derivative ∂ appears as an extra factor in the diagram and, consequently, the real index of divergence d'_Γ (see section 2.2) is necessarily negative.

This means that the operator θ^n requires no counterterms at all, i.e., it is in fact UV finite, $\theta^n = Z[\theta^n]^R$ with $Z = 1$. It then follows that the critical dimension of $\theta^n(x)$ is simply given by expression (2.28) with no correction from γ_F^* and is therefore reduced to the sum of the critical dimensions of the factors:

$$\Delta[\theta^n] = n\Delta[\theta] = n(-1 + \varepsilon/2). \tag{2.31}$$

This justifies the RG equation in the form (2.10) for the correlation functions involving the operators $\theta^n(x)$, in particular, the structure functions (1.1).

Let us turn to the operators

$$F_n \equiv [\partial_i\theta\partial_i\theta]^n \tag{2.32}$$

with $d_F = 0$, $d_F^\omega = -n$. They appear on the left-hand side of equation (1.6) and, as we shall see in section 2.6, it is their critical dimensions that determine the anomalous exponents in (1.4) and (1.6).

In this case, from table 1 we have $d_\Gamma = N_\theta - N_{\mathbf{v}} - (d+1)N_{\theta'}$, with the necessary condition $N_\theta \leqslant 2n$, which follows from the structure of the diagrams. It is also obvious from the analysis of the diagrams that the counterterms to these operators can involve the fields θ, θ' only in the form of derivatives, $\partial\theta$, $\partial\theta'$, and so the real index of divergence has the form $d'_\Gamma = d_\Gamma - N_\theta - N_{\theta'} = -N_{\mathbf{v}} - (d+2)N_{\theta'}$. It then follows that superficial divergences exist only in the Green functions with $N_{\mathbf{v}} = N_{\theta'} = 0$ and any $N_\theta \leqslant 2n$, and the corresponding operator counterterms are reduced to the form F_k with $k \leqslant n$. Therefore, the operators F_n can mix only with each other in renormalization, the corresponding infinite renormalization matrix $Z_F = \{Z_{nk}\}$ is in fact triangular, $Z_{nk} = 0$ for $k > n$, and the critical dimensions associated with the operators F_n are determined by the diagonal elements $Z_n \equiv Z_{nn}$ (in contrast with the case of operators θ^n, they are not equal to unity here).

The constants Z_n are determined by the condition that the 1-irreducible function

$$\Gamma_n = \left\langle F_n^R(x)\theta(x_1)\cdots\theta(x_n)\right\rangle = Z_n^{-1}\langle F_n(x)\theta(x_1)\cdots\theta(x_n)\rangle \tag{2.33}$$

be finite in the renormalized theory. One can write $\Gamma_n = F_n + \sum_{l=1}^\infty \Gamma^{(l)}$, where $\Gamma^{(l)}$ is the sum of the diagrams with l loops with the proper symmetry coefficients. The contributions $\Gamma^{(1)}$ and $\Gamma^{(2)}$, needed for the two-loop calculation of the constant Z_n, are shown in figures 2(a) and (b), respectively. The thick dots correspond to the composite operator F_n, the other diagrammatic elements are the same as in figure 1 (see the text above equation (2.13)). All the

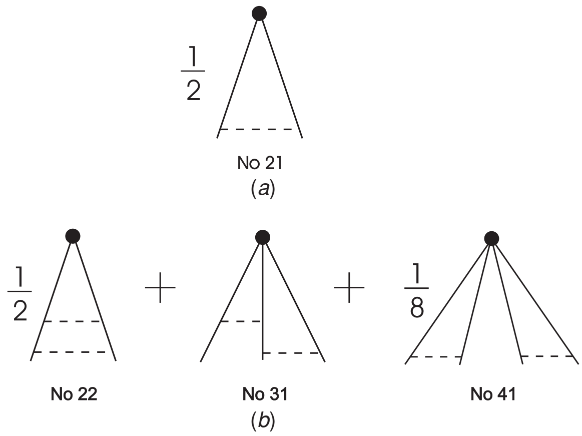

Figure 2. The one-loop (a) and two-loop (b) contributions to the function Γ_n from (2.33).

solid lines correspond to the propagators $\langle\theta\theta'\rangle_0$, the slashes are always placed on the lower ends of the lines, so they are not shown. The nomenclature of the two-loop diagrams is taken from [35]: in 'no MN', M is the number of the solid 'rays' in the diagram and N is the order number of the diagram with given M.

Thus, for the corresponding scaling dimension Δ_n from equation (2.28) and data from table 1, one obtains

$$\Delta_n = n\varepsilon + \gamma_n^*, \qquad \gamma_n = \widetilde{\mathcal{D}}_\mu \ln Z_n. \tag{2.34}$$

In the next section, the dimensions (2.34) will be identified with the anomalous exponents in (1.4). This allows one to construct a systematic perturbation expansion for the latter as series in ε:

$$\Delta_n = \sum_{k=1}^{\infty} \Delta_n^{(k)} \varepsilon^k \tag{2.35}$$

and to use the well-known diagrammatic techniques for the practical calculation of the coefficients $\Delta_n^{(k)}$.

The one-loop calculation gives $\Delta_n^{(1)} = -2n(n-1)/(d+2)$ in agreement with the result (1.5) obtained in [21] within the zero-mode approach. The result $\Delta_1 = 0$ is valid to all orders in ε, which can be proven using certain Schwinger equation, which has the meaning of the energy conservation law [23].

The calculation beyond the one-loop order becomes rather cumbersome and labour consuming and, for Kraichnan's model, was accomplished only in the second [23] and third [35, 36] orders of the ε expansion (two-loop and three-loop approximations, respectively). We refer the interested reader to [36], where the three-loop calculation is presented in detail, and here we give only the answers. The second-order result has the form

$$\Delta_n^{(2)} = \frac{n(n-1)}{(d-1)(d+2)^3(d+4)^2}\{-4(d+1)(d+4)^2 + 3(d-1)(d+2) \\ \times (d+4)(d+2n)h(d) - 4(d+1)(d+2)(d+3n-2)h(d+2)\}, \tag{2.36}$$

where $h(d) \equiv F(1, 1; d/2+2; 1/4)$ and $F(\cdots)$ is the hypergeometric series. Simpler expressions are obtained for integer d, in particular, $h(2) = 8[1 - 3\ln(4/3)]$, $h(3) = 10(\pi\sqrt{3} - 16/3)$, while for the other integer d analogous expressions can be obtained from the recurrent relation $3h(d) + (d+2)h(d+2)/(d+4) = 4$.

No analytical result for $\Delta_n^{(3)}$ is available as a function of d; it can be calculated numerically for any given integer d. In particular,

$$\Delta_n^{(3)} = n(n-2)(0.087\,55n^2 + 0.5192n + 0.2588) \tag{2.37}$$

for $d = 2$ and

$$\Delta_n^{(3)} = n(n-2)(0.0224n^2 + 0.1592n + 0.1372) \tag{2.38}$$

for $d = 3$. The quantities $\Delta_n^{(k)}$ can also be expanded as series in $1/d$; the coefficients of such an expansion can be found analytically, in principle, to any given order. To the order ε^3 with the accuracy of $1/d^2$, one has

$$\Delta_n = \varepsilon n(n-2)\{-(1-2/d)/2d + 3\varepsilon/4d^2 + 7\varepsilon^2/4d^2\}. \tag{2.39}$$

Note that expressions (2.36) and (2.39) are in agreement with the $O(1/d)$ result for Δ_n obtained in [22]; see equation (1.5). The latter shows that the $O(1/d)$ contribution is completely contained in the $O(\varepsilon)$ term, while the higher order terms $O(\varepsilon^k)$ with $k \geqslant 2$ for large d involve no $O(1/d)$ terms and behave as $O(1/d^2)$.

The knowledge of the three terms of the ε expansion in the model (2.1)–(2.3) allows one to discuss its convergence properties and to obtain improved predictions for finite ε in reasonable agreement with the existing nonperturbative results: analytical solution of the zero-mode equations for $n = 2$ [22], numerical solutions for $n = 3$ [58] and numerical experiments for $n = 4$ [59] and $n = 6$ [60].

2.6. Operator product expansion and anomalous scaling

As already noted, representations of the types (2.22)–(2.24) and (2.27) for any scaling functions $\xi(mr)$ describe the behaviour of the correlation functions for $\mu r \sim \Lambda r \gg 1$ and any fixed value of mr. The inertial range corresponds to the additional condition $mr \ll 1$. The form of the functions $\xi(mr)$ is not determined by the RG equations themselves; they can be calculated as series in ε, but this is useless for the analysis of their behaviour at $mr \to 0$. By analogy with the theory of critical phenomena [7, 8], it can be studied using the operator product expansion. Below we concentrate on the equal-time structure functions (1.1) and (2.24).

According to the OPE, the behaviour of the quantities entering into the right-hand side of equation (1.1) for $\mathbf{r} = \mathbf{x} - \mathbf{x}' \to 0$ and fixed $\mathbf{x} + \mathbf{x}'$ is given by the infinite sum

$$[\theta(t,\mathbf{x}) - \theta(t,\mathbf{x}')]^n = \sum_F C_F(\mathbf{r}) F\left(t, \frac{\mathbf{x}+\mathbf{x}'}{2}\right), \tag{2.40}$$

where C_F are coefficients regular in m^2 and F are all possible renormalized local composite operators allowed by the symmetry. More precisely, the operators entering into the OPE are those which appear in the naive Taylor expansion and all the operators that admix to them in renormalization.

In what follows, it is assumed that the expansion is made in irreducible tensors (scalars, vectors and traceless tensors); the possible tensor indices of the operators F are contracted with the corresponding indices of the coefficients C_F. With no loss of generality, it can also be assumed that the expansion is made in 'scaling' operators, i.e., those having definite critical dimensions Δ_F (see section 2.5).

The structure functions (1.1) are obtained by averaging equation (2.40) with the weight $\exp \mathcal{S}_R$ with the renormalized action (2.7); the mean values $\langle F\rangle$ appear on the right-hand side. Their asymptotic behaviour for $m \to 0$ is found from the corresponding RG equations and has the form $\langle F\rangle \propto m^{\Delta_F}$.

From the RG representation (2.24) and the operator product expansion (2.40), we therefore find the following expression for the structure function in the inertial range ($\Lambda r \gg 1$, $mr \ll 1$):

$$S_n(\mathbf{r}) = D_0^{-n/2} r^{n(1-\varepsilon/2)} \sum_F A_F(m\mathbf{r})(mr)^{\Delta_F}, \tag{2.41}$$

where the coefficients A_F are regular in $(mr)^2$.

Some general remarks are now in order.

Owing to the translational invariance, the operators having the form of total derivatives give no contribution to equation (2.41): $\langle \partial F(x)\rangle = \partial \langle F(x)\rangle = \partial \times \text{const} = 0$.

In the model (2.1)–(2.3), the operators with an odd number of fields θ also have vanishing mean values; their contributions vanish along with the odd structure functions themselves (they will be 'activated' in the presence of a nonzero mixed correlation function $\langle \mathbf{v} f\rangle$; we shall not discuss this possibility here).

Owing to the isotropy of the model, only contributions of the scalar operators survive in (2.41). Indeed, in the $SO(d)$-covariant case, the mean value of a tensor operator depends only on scalar parameters, its tensor indices can only be those of Kronecker delta symbols. It is impossible, however, to construct nonzero irreducible (traceless) tensor solely of the delta symbols. The coefficients A_F and therefore the functions S_{2n} depend only on $r = |\mathbf{r}|$. (In the presence of anisotropy, irreducible tensor operators acquire nonzero mean values and their contributions appear on the right-hand side of (2.41). Such models will be discussed in section 3.)

The feature characteristic of the models describing turbulence is the existence of the so-called 'dangerous' composite operators with *negative* critical dimensions; see [31, 32]. Their contributions into the OPE give rise to singular behaviour of the scaling functions for $mr \to 0$, that is, the anomalous scaling.

Since $\Delta_F = d_F + O(\varepsilon)$, the operators with minimal Δ_F are those involving maximum possible number of fields θ and minimum possible number of derivatives (at least for small ε). Both the problem (2.1)–(2.3) and the quantities (1.1) possess the symmetry $\theta \to \theta + \text{const}$. It then follows that the expansion (2.40) involves only operators invariant with respect to this shift and therefore built of the *gradients* of θ.

As already mentioned, the operators entering into the right-hand side of equation (2.40) are those which appear in the Taylor expansion and those that admix to them in renormalization. The leading term of the Taylor expansion for S_n is the $2n$th rank operator which can symbolically be written as $(\partial\theta)^n$; its decomposition in irreducible tensors gives rise to operators of lower ranks. These contributions exist in the OPE (before averaging) even if the stirring force in not included into equation (2.1); in the language of [21, 22], it is then tempting to identify them with the zero modes. In the presence of the stirring force, operators of the form $(\partial\theta)^k$ with $k < n$ admix to them in renormalization and appear in the OPE; their contributions correspond to solutions of the inhomogeneous equations. Owing to the linearity of problem (2.1), operators with $k > n$ (whose contributions would be more important) do not admix in renormalization to the terms of the Taylor expansion for S_n and do not appear in the corresponding OPE. All these operators have minimal possible canonical dimension $d_F = 0$ (see table 1) and determine the leading terms of the $mr \to 0$ behaviour in the sectors with $j \leqslant 2n$. Operators involving more derivatives than fields θ (and thus having canonical dimensions $d_F = 1, 2$ and so on) determine correction terms for $mr \to 0$.

We thus have established the representations (1.4) within the RG and OPE approach and identified the anomalous exponents Δ_n with the scaling dimensions of the composite fields $F_n \equiv [\partial_i\theta\partial_i\theta]^n$.

Representations analogous to (1.4) can be written for other quantities. In particular, for the equal-time pair correlation functions of the operators $E^n(x) = \kappa_0^n F_n(x)$ with F_n from (2.32), powers of the dissipation rate of scalar fluctuations, one obtains

$$\begin{aligned} \langle E^n(x) E^p(x') \rangle &= (\Lambda r)^{-\Delta_n - \Delta_p} f_{n,p}(mr), \\ f_{n,p}(mr) &= \mathrm{const}(mr)^{\Delta_{n+p}} \end{aligned} \tag{2.42}$$

with the same dimensions Δ_n as in (1.4). The first expression follows from the corresponding RG equation and holds for $(\Lambda r) \gg 1$ and arbitrary fixed (mr); the second follows from the corresponding OPE with the leading term determined by the operator F_{n+p} and holds for $(mr) \ll 1$. More examples, e.g. those with tensor composite operators, can be found in [22, 23]. Note that for $n = p = 1$ the correlation function (2.42) is independent of the UV scale Λ, which can be viewed as the analogue of the second Kolmogorov hypothesis. It is also worth noting that the family of operators $E^n \propto F_n$ is 'closed with respect to the fusion' in the sense that the leading term in the OPE for the pair correlator (2.42) is given by the operator F_{n+m} from the same family with the summed index $n + m$. This fact along with the inequality $\Delta_n + \Delta_m > \Delta_{n+m}$, which is obvious from the explicit expressions for Δ_n, can be interpreted as the statement that the correlations of the local dissipation rate in the model (2.1)–(2.3) exhibit multifractal behaviour, see [25, 26]. It remains to note that the same inequality ensures the fulfilment of the Hölder inequality for the structure functions (1.4).

2.7. Comparison with numerical experiments

The knowledge of the three terms of the ε expansion for the anomalous exponents Δ_n in the model (2.1)–(2.3) allows one to discuss the nature and convergence properties of such expansions in dynamical models in general, to try to construct improved expansions and to obtain reasonable predictions for finite $\varepsilon \sim 1$. An important advantage of the Kraichnan model is the existence of exact analytical results and very accurate numerical simulations: analytical solution of the zero-mode equations for $n = 2$ [22], numerical solutions for $n = 3$ [58] and numerical experiments for $n = 4$ [59] and $n = 6$ [60]. This gives a unique opportunity to compare those nonperturbative data with the perturbative results of the ε expansion. In some respects, such a comparison is more interesting than that with actual experiments, because all the results pertain to the same exactly defined model, while in experiments any deviations from theoretical predictions can be attributed to the effect of impurities, unaccounted interactions and so on[1]. These issues are discussed in [35, 36].

Due to the time decorrelation of the velocity field, in the rapid-change model the Eulerian and Lagrangian statistics of the velocity field are identical. This allows one to perform very accurate numerical simulations in the Lagrangian frame, because it is sufficient to generate the velocity field only along the particles' trajectories. In practice, the functions S_4 [59] and S_6 [60] were determined for various values of the parameter ε.

As far as the ε dependence is concerned, the simulations [59, 60] show that Δ_n decreases with ε, achieves a minimum at some point inside the interval $0 < \varepsilon < 2$ and then increases to zero at $\varepsilon = 2$; the depth of the minimum grows and its position moves to the origin as n grows from 4 to 6 or d decreases from 3 to 2. It turns out that even the naive sum of the first three terms of the ε expansion reproduce these, rather subtle, features of the nonperturbative results. This behaviour is illustrated by figures 3(*a*) ($n = 6$) and (*b*) ($n = 4$), where the results of the improved ε expansion (see below) are also shown.

[1] It is worth noting that, for $d = 3$ and the most realistic (Kolmogorov) value $\varepsilon = 4/3$, the results of [59, 60] are in a surprisingly good agreement with known experimental estimates for the anomalous exponents (e.g. [12, 13]), which supports the relevance of the Kraichnan model for description of the real turbulent transfer.

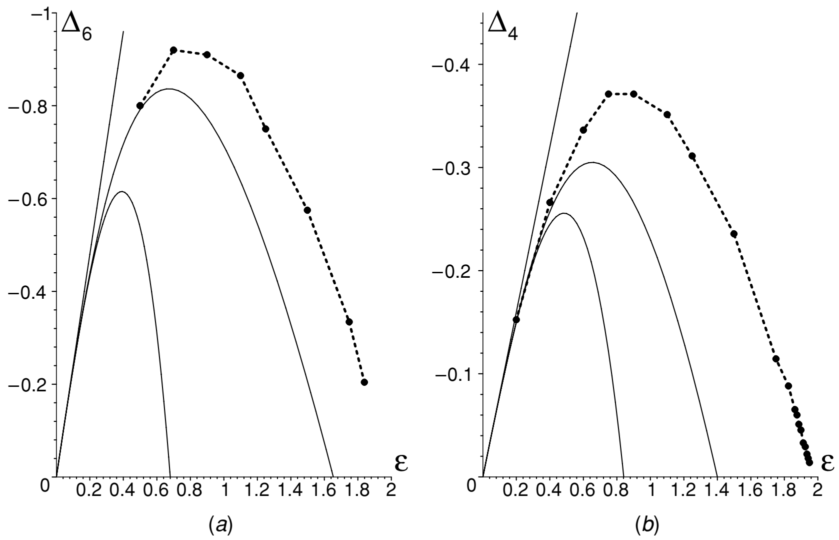

Figure 3. Dimensions Δ_n for $d = 3$: $n = 6$ (*a*) and $n = 4$ (*b*). Dots connected by dashed lines: numerical simulations by [60] ($n = 6$) and [59] ($n = 4$). Solid lines: the $O(\varepsilon)$ slope, third-order approximation of the improved ε expansion and third-order approximation of the plain ε expansion (from above to below).

For small ε, the agreement between the ε expansion and nonperturbative results improves when the higher order terms are taken into account, but the deviation becomes remarkable for $\varepsilon \sim 1$ and decreasing d. Furthermore, the convergence of the ε series appears more irregular for $d = 2$, while the forms of the nonperturbative results are not much affected by the choice of d. Such behaviour can be naturally explained by exact analytical results for the exponents describing anisotropic contributions to S_2 (they are nontrivial in the presence of the large-scale anisotropy; see section 3.2). They suggest that in the rapid-change model the series in ε have finite radii of convergence ε_c, depending on the exponent in question and ranging from 0 to ∞ when d varies from 1 to ∞. Hence, the direct summation of the ε expansion for Δ_n works only in the interval $\varepsilon < \varepsilon_c$, which decreases almost linearly with $(d - 1)$.

The difference with the models of critical phenomena, where ε series are always asymptotical, can be traced back to the fact that in the rapid-change models there is no factorial growth of the number of diagrams in higher orders of the perturbation theory: too many diagrams vanish due to the presence of the closed circuits of retarded propagators (see item (2) in section 2.2). The divergence of ε_c for $d \to 1$ can be related to the fact that the transverse vector field ceases to exist in one dimension. Therefore, in order to improve the convergence and to obtain reasonable predictions for finite ε, one should augment plain ε expansions by the information about the character of the singularities and their location in the complex ε plane, which can be extracted from the asymptotical behaviour of the coefficients $\Delta_n^{(k)}$ in equation (2.35) at large k.

In field-theoretic models, such behaviour is studied with the aid of instanton analysis (steepest descent calculation of the relevant functional integrals); see e.g. [7]. No such information, however, is available yet for the exponents Δ_n in the model (2.1)–(2.3). The instanton analysis of [61] did not touch upon the problem of the large-order coefficients of perturbative series. It has mostly been concentrated on the behaviour of the exponents at large n and predicts saturation (that is, existence of finite limit) of the total exponent $n(2 - \varepsilon) + \Delta_n$

in (1.4) for $n \to \infty$. In [62], the instanton method was applied to the scaling dimensions of composite operators θ^n. In contrast to the case (2.3), where they are given exactly by simple relations (2.31), they become infinite series in ε if the advecting velocity field is not transverse (see section 3.1). The analysis of [62] indeed demonstrates the absence of factorial growth in the large-k behaviour. Unfortunately, no generalization to operators built of derivatives, and hence to the dimensions Δ_n which we are interested in here, has been obtained yet.

It turns out, however, that certain elementary considerations allow one to improve the ε expansions without the information about their higher order behaviour [36]. Assume that the ε series for γ_n^* has a finite radius of convergence, and that the singularity that determines it (closest to the origin) is algebraic. Then this singularity will disappear for the 'inverse ε expansion'—ε as a series in γ_n^*—and the convergence of the latter improves in comparison to the direct one. Practical implementation of this idea leads to an apparent improvement of the convergence of the ε series and, at the same time, to a better agreement with the nonperturbative results. Further improvement can be achieved using an interpolation formula which combines the first terms of the ε expansion with the asymptotic form of the dimensions Δ_n in the opposite limit $\varepsilon \to 2$, proposed (for $n = 4$) in [59] on the basis of their numerical simulation.

The conclusion is optimistic: the first few terms of the ε expansion, augmented by simple additional considerations, provide an adequate description of the anomalous behaviour also for finite values of ε.

3. Aspects of universality: effects of compressibility, anisotropy and memory

In the original Kraichnan model (2.1)–(2.3), the velocity field is taken to be Gaussian, isotropic, incompressible and decorrelated in time. Of course, such assumptions are strong departures from the statistical properties of genuine turbulence. More realistic models should involve anisotropy, compressibility, finite correlation time and so on. Recent studies have pointed up some significant differences between the zero and finite correlation-time problems [63, 64] and between the compressible and incompressible cases [65–70].

In [71], a generalized phenomenological model was considered in which the temporal correlation of the advecting field was set by eddy turnover. It was argued that the anomalous exponents may depend on more details of the velocity statistics, than just the exponents. This idea has received further analytical support in [63, 72], where the case of short but finite correlation time was considered for the special case of a local turnover exponent. The anomalous exponents were derived to first order in small correlation time, with Kraichnan's rapid-change model [63] or analogous shell model for a scalar field [72] taken as zeroth-order approximations. The exponents obtained appear nonuniversal through the dependence on the correlation time. Various aspects of the transport and dispersion of particles in random Gaussian self-similar velocity fields with finite correlation time were also studied in [73–76].

Another important question recently addressed is the effects of large-scale anisotropy on inertial-range statistics of passively advected fields [14–16, 41, 48, 49, 51, 58, 69] and the velocity itself [77]. These studies have shown that the anisotropy present at large scales has a strong influence on the small-scale statistical properties of the scalar, in disagreement with what was expected on the basis of the cascade ideas [14–16, 58, 69]. On the other hand, the exponents describing the inertial-range scaling exhibit universality and hierarchy related to the degree of anisotropy, which gives some quantitative support to Kolmogorov's hypothesis on the restored local isotropy of the inertial-range turbulence [48, 49, 51, 77].

In a wider context, the Kraichnan model and its descendants can be interesting as model systems for studying generic nonequilibrium dynamical features. Recently, significant progress has been achieved in classifying large-scale, long-distance scaling behaviour of such systems, including driven diffusive systems [78], diffusion-limited reactions [79], ageing, growth and percolation processes [80], and so on. Being analytically tractable, Kraichnan-type models can serve as a unique testing ground in studying such scaling regimes and their universality in general.

3.1. Effects of compressibility

There are two types of diffusion–advection problems for the compressible velocity field. Passive advection of a density field (say, the density of an impurity) is described by the equation

$$\partial_t\theta + \partial_i(v_i\theta) = \kappa_0\partial^2\theta + f, \tag{3.1}$$

while the advection of a 'tracer' (say, temperature, specific entropy or *concentration* of the impurity particles) is described by

$$\partial_t\theta + (v_i\partial_i)\theta = \kappa_0\partial^2\theta + f. \tag{3.2}$$

In the rapid-change model, f is a Gaussian noise with correlator (2.2), while the correlator of the Gaussian velocity field is given by expression (2.3) with the replacement

$$D_0P_{ij}(\mathbf{k}) \to D_0P_{ij}(\mathbf{k}) + D_0'Q_{ij}(\mathbf{k}) = D_0(P_{ij}(\mathbf{k}) + \alpha Q_{ij}(\mathbf{k})), \tag{3.3}$$

where $Q_{ij}(\mathbf{k}) = k_ik_j/k^2$ is the longitudinal projector and $D_0' > 0$ is an additional amplitude factor. The case $\alpha = 0$ corresponds to the purely transverse velocity field, when $\partial_iv_i = 0$ and the models (3.1) and (3.2) coincide. The opposite limit $\alpha \to \infty$ at fixed $D_0' = D_0\alpha$ corresponds to a purely potential velocity. As a measure of the degree of compressibility, one often uses the parameter $\wp = \alpha/(d-1+\alpha)$ which satisfies the inequalities $0 \leqslant \wp \leqslant 1$.

It was shown in [68–70] that the anomalous scaling regime in the compressible models breaks down if both ε and α are large enough (namely, for $\wp \equiv \alpha/(d-1+\alpha) > d/\varepsilon^2$) and the inverse energy cascade with no anomalous scaling takes place. However, in the perturbative regions (small ε or $1/d$) this effect does not take place, and the anomalous scaling can be studied using the ε expansion.

The RG analysis of sections 2.2 and 2.3 is directly extended to the case $\alpha \neq 0$. The field-theoretic analogues of the stochastic models (3.1) and (3.2) are multiplicatively renormalizable with the only independent renormalization constant Z_κ, and the corresponding RG equations possess an IR-stable fixed point. Its coordinate is the same for both the models and has the form

$$g_* = \frac{2d\varepsilon}{C_d(d-1+\alpha)} \tag{3.4}$$

with C_d from (2.14). The fixed point is degenerate in the sense that g_* depends on the parameter α. From the RG viewpoints, α can be treated as the second coupling constant. The corresponding β function $\beta_\alpha \equiv \widetilde{\mathcal{D}}_\mu\alpha$ vanishes identically owing to the fact that α is not renormalized. Therefore, the equation $\beta_\alpha = 0$ gives no additional constraint on the values of the parameters g, α at the fixed point. The value of $\gamma_\nu(g)$ at the fixed point is independent of α and is exactly given by the same expression (2.17).

The tracer field enters equation (3.2) only in the form of a derivative, so that the invariance with respect to the shift $\theta \to \theta + \text{const}$ holds in this model for all α. Therefore, the analysis of the composite operators and operator product expansions, performed in sections 2.5 and

2.6 for the case $\alpha = 0$, applies to the tracer model for general α. The operators θ^n are not renormalized, their dimensions are independent of α and coincide with equation (2.31), the structure functions satisfy the ordinary RG equations (2.10) and their inertial-range behaviour is given by expressions (1.4) with the anomalous exponents Δ_n determined by the scaling dimensions of the operators (2.32). The only difference is that in the present model these dimensions depend on α. In the first order in ε, one has

$$\Delta_n = -2n(n-1)\varepsilon(1+2\wp)/(d+2) + O(\varepsilon^2) \tag{3.5}$$

with $\wp = \alpha/(d-1+\alpha)$, that is, the dimension Δ_n for $\alpha = \wp = 0$ is multiplied by the factor $(1+2\wp) \geqslant 1$. This result was obtained using the zero-mode technique in [68, 69], where the qualitative observation was made that the intermittency increases with the degree of compressibility. The $O(\varepsilon^2)$ correction (with more cumbersome dependence on α) was derived later in [38] using the RG and OPE approach.

The case of the density field appears rather different [37]. Obviously, the field θ enters equation (3.1) not only in the form of a derivative, but also without it. As a result, the invariance with respect to the shift $\theta \to \theta + \text{const}$ does not hold for any $\alpha \neq 0$. Although the composite operators θ^n remain multiplicatively renormalizable (see the discussion in section 2.5), they require nontrivial renormalization, $\theta^n = \bar{Z}_n[\theta^n]^R$ with $\bar{Z}_n \neq 1$, and, therefore, they acquire nontrivial scaling dimensions $\bar{\Delta}_n$ different from the simple multiplies of Δ_θ in (2.31). To the second order, one obtains [37]

$$\begin{aligned}\bar{\Delta}_n = n(-1+\varepsilon/2) - \frac{\alpha n(n-1)d\varepsilon}{2(d-1+\alpha)} + \frac{\alpha(\alpha-1)n(n-1)(d-1)\varepsilon^2}{2(d-1+\alpha)^2} \\ + \frac{\alpha^2 n(n-1)(n-2)dh(d)\varepsilon^2}{4(d-1+\alpha)^2} + O(\varepsilon^3),\end{aligned} \tag{3.6}$$

with $h(d) \equiv F(1, 1; d/2+1; 1/4)$.

As a consequence, different terms in the decomposition

$$S_n = \sum_{k+p=n} C_n^{kp}\langle\theta^k(x)\theta^p(x')\rangle$$

acquire different scaling behaviours, with the leading contribution being given by the constant $\langle\theta^n\rangle$. Thus, the anomalous scaling now reveals itself not in the structure functions, but in the pair correlation functions of the composite fields θ^n:

$$\begin{aligned}&\langle\theta^n(x)\theta^p(x')\rangle = (\kappa_0\Lambda^2)^{-(n+p)/2}(\Lambda r)^{-\bar{\Delta}_n-\bar{\Delta}_p} f_{n,p}(mr), \\ &f_{n,p}(mr) = \text{const}(mr)^{\bar{\Delta}_{n+p}},\end{aligned} \tag{3.7}$$

with the dimensions $\bar{\Delta}_n$ from (3.6). As usual, the first expression follows from the corresponding RG equation and holds for $(\Lambda r) \gg 1$ and arbitrary fixed (mr); the second follows from the OPE with the leading term determined by the operator $\theta^{(n+p)}$ (without derivatives) and holds for $(mr) \ll 1$. For $d = 1$, expression (3.7) agrees with the result derived earlier in [67] within the zero-mode approach. Generalization to the cases of finite correlation time and presence of large-scale anisotropy were studied in [38, 53].

3.2. Large-scale anisotropy

In real experiments or numerical simulations, anisotropy emerges in the system due to nontrivial geometry of the boundaries, obstacles or stirring devices, and is therefore introduced at large scales of order L. In the works on the Kraichnan model, it is usually modelled by assigning anisotropic properties to the correlation function of the random noise. In general, instead of (2.2) it is taken in the form

$$\langle f(x)f(x')\rangle = \delta(t-t')C(\mathbf{r}/L), \qquad C(\mathbf{r}/L) = \sum_{lm} C_{lm}(r/L)Y_{lm}(\mathbf{n}), \tag{3.8}$$

where $C_{lm}(r/L)$ are coefficient functions finite at $r = 0$ and rapidly decaying for $(r/L) \to \infty$ and $Y_{lm}(\mathbf{n})$ are the spherical harmonics carrying the angular dependence. In the special case of uniaxial anisotropy (which is often sufficient to reveal all the new anomalous exponents), only terms with $m = 0$ (Legendre polynomials for $d = 3$ or Gegenbauer polynomials for general d) enter the right-hand side of (3.8). The anisotropy makes it possible to also introduce a mixed correlator $\langle \mathbf{v} f\rangle$, which gives rise to nonvanishing odd correlation functions of θ but leads to no serious alterations in the analysis. Another possibility is to substitute $\theta(x) \to (\mathbf{hx}) + \theta(x)$ with a constant vector $\mathbf{h}$, which produces in equation (2.1) the term $\mathbf{hv}$ which replaces the artificial random noise, maintains the steady state and also gives rise to nonvanishing even and odd correlation functions of θ. This imposed constant gradient mimics, e.g., a constant temperature difference between the distant walls [15, 16]. The representation analogous to (3.8) can be written for the relevant correlation functions. In the following, we restrict ourselves with the special case of uniaxial anisotropy, then the structure functions (1.1) are written in the form

$$S_n(\mathbf{r}) = \sum_l S_{nl}(r)P_l(\mathbf{n}), \tag{3.9}$$

where P_l are the Gegenbauer (Legendre for $d = 3$) polynomials, S_{nl} are scalar coefficient functions dependent only on $r = |\mathbf{r}|$ and the functions S_n, in general, do not vanish for odd n.

According to the classical hypothesis on the local isotropy restoration [2], the anisotropy introduced at large scales dies out when the energy is transferred down to smaller scales (inertial-range) owing to the cascade mechanism. In most works on Kraichnan's model, this hypothesis is tacitly accepted for the statistics of the velocity field, and the latter is still described by the isotropic ensemble (2.3). Then, for even structure functions, both the zero-mode and the RG approach give the following *hierarchical* picture of the isotropy restoration: in the inertial range, the coefficient functions S_{nl} reveal a power-law behaviour,

$$S_{nl}(r) \simeq s_{nl}(r/L)^{\zeta_{nl}}, \tag{3.10}$$

with nonuniversal (e.g. dependent on the coefficients in the decomposition (3.8), the form of the IR cut-off in (2.3), etc) and universal exponents ζ_{nl} dependent only on ε and d, but not on the way the anisotropy was introduced. The exponents exhibit a kind of hierarchy related to the degree of anisotropy: for a given n, the higher is the order l of the 'anisotropic shell', the larger is the exponent ζ_{nl} and, therefore, the less important is its contribution for $(r/L) \ll 1$:

$$\zeta_{nl} > \zeta_{nl'} \qquad \text{for} \quad l > l'. \tag{3.11}$$

Thus, the leading contribution in the inertial-range behaviour is given by the isotropic shell ($l = 0$), the corresponding exponent is the same as for the purely isotropic model (2.2), while the anisotropic contributions give only corrections which vanish for $(r/L) \to 0$, the decay becomes faster when the order $l \geqslant 1$ increases.

For the passive scalar field and incompressible velocity ensemble (model (2.1), (2.3)), the exponents ζ_{2l} for the second-order function S_2 were derived exactly (that is, without expansion in ε and for general d) in [22] within the zero-mode approach (see also [81] for $d = 2$ and 3). For general n, the exponents ζ_{nl} were derived, using the RG and OPE approach and in the one-loop approximation, in [51] for the velocity ensemble with finite correlation time; the two-loop result was derived in [52]. For the special case of the Kraichnan model (2.3), the one-loop result was reproduced later in [46, 82]. For the passive vector (magnetic) field, exact answers for $n = 2$ and general l were derived in [48], and the $O(\varepsilon)$ result for general n, l can be found in [41].

Let us briefly outline the RG derivation [51] of the exponents (3.11) for the simplest case of a scalar field and vanishing correlation time (2.1), (2.3), (3.8). The RG analysis of section 2 remains essentially the same. In particular, the renormalization constants (2.8), the coordinate of the fixed point (2.16) and the basic critical dimensions in (2.23) and (2.31) remain the same, because the diagrams needed for their calculation do not involve the correlation function of the noise. The main difference is that for isotropic case, only scalar operators contribute to the representations like (2.41), while in the presence of anisotropy the irreducible tensor operators acquire nonzero mean values and also contribute to equation (2.41). For example, the mean value of the operator $\partial_i\theta\partial_j\theta - \delta_{ij}\partial_k\theta\partial_k\theta/d$ in the case of uniaxial anisotropy is proportional to the irreducible tensor $n_i n_j - \delta_{ij}/d$ built of the vector $\mathbf{n}$ that specifies the preferred direction. Contraction of such tensors with the Wilson coefficients $C_F(\mathbf{r})$ in (2.40) gives rise to contributions proportional to $P_l(\mathbf{n})$ in representation (2.41), with the order l equal to the rank of the operator. Thus, the exponents ζ_{nl} in (3.10) are identified as $\zeta_{nl} = n(1 - \varepsilon/2) + \Delta_{nl}$, where the first term comes from the RG representation (2.22) and Δ_{nl} is the *minimal* dimension (2.28) of a lth rank tensor composite operator that can give contribution to the OPE in question. From the analysis of the dimensions, it easily follows that such an operator should involve minimal possible number of derivatives ∂ and maximal possible number of fields θ. Thus, for general n and $l \leqslant n$, the set dimensions Δ_{nl} are determined by the family of operators

$$F_{n,l} \equiv \partial_{i_1}\theta \cdots \partial_{i_l}\theta(\partial_i\theta\partial_i\theta)^p + \cdots, \tag{3.12}$$

where l is the number of the free vector indices and $n = l + 2p$ is the total number of the fields θ entering into the operator. The vector indices of the symbol $F_{n,l}$ are omitted; the dots stand for subtracted terms with Kronecker delta symbols which make the tensor (3.12) irreducible.

Note that $F_{n,0} = F_n$ from (2.32). Although the operators (3.12) mix in renormalization, the corresponding matrix Z appears triangular, the dimensions $\Delta_{n,l}$ are determined by its diagonal elements and thus can be identified as dimensions of individual monomials $F_{n,l}$. Furthermore, they are determined by the 1-irreducible functions $\langle F_{n,l}(x)\theta(x_1)\cdots\theta(x_n)\rangle_{\text{1-ir}}$ (see section 2.5), which do not involve the correlator (3.8); hence the independence on the noise and the coincidence of the dimension $\Delta_{n,0}$ with Δ_n from (2.34). More detailed discussion of these issues can be found e.g. in [51, 53]. In the one-loop approximation, one obtains [51]

$$\zeta_{n,l} = n + \frac{2n(n-1) - (d+1)(n-l)(d+n+l-2)}{2(d-1)(d+2)}\varepsilon, \tag{3.13}$$

which for $l = 0$ agrees with (1.5); the $O(\varepsilon^2)$ term was derived in [52]. The inequalities (3.11) simply follow from the explicit expression (3.13).

The generalization of the result (3.13) to the case of compressible velocity ensemble (with vanishing and finite correlation time) was presented in [51]:

$$\zeta_{n,l} = n + \frac{2n(n-1)(1-\alpha) - (d+1+\alpha)(n-l)(n+l+d-2)}{2(d+2)(d-1+\alpha)}\varepsilon. \tag{3.14}$$

Detailed derivation was given in [53]; the $O(\varepsilon^2)$ contribution and exact results for $n = 2$ (for Kraichnan's case) can be found in [52]. Passive magnetic fields were studied in [42]. The main qualitative conclusion which can be drawn from the explicit expression (3.14), its analogue for magnetic fields and exact results for $n = 2$ is that, although the hierarchy relations (3.11) remain valid for all values of $\alpha > 0$, the corrections become closer to leading terms as α increases. This statement can be expressed in the inequalities

$$\partial\Delta_{n,l}/\partial l > 0, \qquad \partial^2\Delta_{n,l}/\partial l\partial\alpha < 0. \tag{3.15}$$

The hierarchical inequalities and analytical results discussed above support and refine the classical phenomenological hypothesis on the local isotropy restoration for passively advected scalar and vector fields. However, the general hierarchical picture (superpositions of power laws with universal exponents and nonuniversal amplitudes) seems much more general, being compatible with that established recently in the field of NS turbulence, on the basis of numerical simulations of channel flows and experiments in the atmospheric surface layer, see [77] and references therein. In those papers, the velocity structure functions were decomposed into the irreducible representations of the rotation group. It was shown that in each sector of the decomposition, scaling behaviour can be found with apparently universal exponents. The amplitudes of the various contributions are nonuniversal, through the dependence on the position in the flow, the local degree of anisotropy and inhomogeneity, and so on [77].

Nevertheless, the anisotropy survives in the inertial range and reveals itself in *odd* correlation functions [15, 16, 58, 53]. Consider the odd-order dimensionless ratios

$$\mathcal{R}_{2n+1} \equiv S_{2n+1} \big/ S_2^{n+1/2} \propto (mr)^{\Delta_{2n+1,1}-(n+1/2)\Delta_{2,0}}, \tag{3.16}$$

where $\mathcal{R}_3$ is the skewness factor. The last relation, valid in the inertial range of scales, follows from the RG and OPE representations (section 2) and the observation that the leading contribution to the OPE of an odd structure function S_{2n+1} is given by the vector operator $F_{2n+1,1}$ from (3.12). Substituting explicit one-loop expressions for the dimensions gives (see e.g. [53])

$$\mathcal{R}_{2n+1} \propto (mr)^{\varepsilon[(d-1+\alpha)(d+2-4n^2)-8\alpha n^2]/2(d+2)(d-1+\alpha)}. \tag{3.17}$$

According to the naive cascade ideas, the quantities (3.16) were expected to decrease for $mr \to 0$. From (3.17) it follows that, for $\alpha = 0$, $\mathcal{R}_3$ decreases but slower than expected on the basis of the cascade picture (the result obtained within the zero-mode approach in [58]), while the higher order ratios diverge as $mr \to 0$. The latter fact agrees with the findings of [83], where the passive advection by the two-dimensional Navier–Stokes velocity field was studied in a numerical experiment. For general α, already $\mathcal{R}_3$ becomes divergent for $mr \to 0$ provided the compressibility is strong enough (namely, if $\alpha > (d-1)(d-2)/(10-d) + O(\varepsilon)$), while the divergence of the higher order ratios becomes even faster as α increases.

This means that compressibility enhances the penetration of the large-scale anisotropy towards the depth of the inertial range. This fact also seems universal, being observed in the model of the passively advected magnetic field [42].

The case of anisotropic velocity ensemble, where the strong anisotropy persists to all scales, was also studied for the passive scalar [39] and vector magnetic [40] fields. In these studies, the ordinary transverse projector $P_{ij}(\mathbf{k})$ in equation (2.3) was replaced with the general transverse structure that possesses the uniaxial anisotropy:

$$T_{ij}(\mathbf{k}) = a(\psi) P_{ij}(\mathbf{k}) + b(\psi) \tilde{n}_i(\mathbf{k}) \tilde{n}_j(\mathbf{k}). \tag{3.18}$$

Here the unit vector $\mathbf{n}$ determines the distinguished direction ($\mathbf{n}^2 = 1$), $\tilde{n}_i(\mathbf{k}) \equiv P_{ij}(\mathbf{k}) n_j$ and ψ is the angle between the vectors $\mathbf{k}$ and $\mathbf{n}$, so that $(\mathbf{nk}) = k\cos\psi$ (note that $(\tilde{\mathbf{n}}\mathbf{k}) = 0$). The scalar functions can be decomposed in the Gegenbauer polynomials (the d-dimensional generalization of the Legendre polynomials):

$$a(\psi) = \sum_{l=0}^{\infty} a_l P_{2l}(\cos\psi), \qquad b(\psi) = \sum_{l=0}^{\infty} b_l P_{2l}(\cos\psi). \tag{3.19}$$

The positivity of the correlator (2.3) leads to the conditions

$$a(\psi) > 0, \qquad a(\psi) + b(\psi)\sin^2\psi > 0. \tag{3.20}$$

In practical calculations, the authors of [39, 40] mostly confined themselves with the special case

$$T_{ij}(\mathbf{k}) = (1 + \rho_1 \cos^2 \psi) P_{ij}(\mathbf{k}) + \rho_2 \tilde{n}_i(\mathbf{k}) \tilde{n}_j(\mathbf{k}), \tag{3.21}$$

when the inequalities (3.20) reduce to $\rho_{1,2} > -1$. This case represents nicely the typical features of the general model (3.18).

The RG and OPE approach is applicable to this model; the RG equations possess an IR-stable fixed point whose coordinate depends on the anisotropy parameters $\rho_{1,2}$ (like it depends on α for the compressible case; see section 3.1). As a result, the anomalous exponents, which are still determined by the scaling dimensions related to the family (3.12), become nonuniversal through the dependence on $\rho_{1,2}$. The key difference with the case of large-scale anisotropy is that for the model (3.18), the operators (3.12) with equal n and different l mix heavily in renormalization: the matrix $Z_{l,l'}$ in the renormalization relation $F_{n,l} = Z_{l,l'} F^R_{n,l'}$ is neither diagonal nor triangular, and it cannot be made triangular by changing to irreducible tensor operators. Now the small-(mr) behaviour of the scaling functions $\xi(mr)$ in (2.24) is determined by the *eigenvalues* of the matrix Δ_F from (2.28); the minimal eigenvalue (which is no longer identified with the scaling dimension of an individual, e.g. scalar, composite operator F_{nl}) determines the leading term in the representation (2.41), while the other determines the corrections. No explicit analytical expression for general n and l, analogous to (3.13), is available for finite $\rho_{1,2}$: although the matrices Δ_F were calculated analytically as a function of $\rho_{1,2}$ and d, the diagonalization can only be performed numerically, separately for different families with given n, and the results can be represented graphically as functions of $\rho_{1,2}$ and fixed d. Analytical results can be derived only within the expansion in $\rho_{1,2}$.

In principle, the nontrivial structure of the matrices $Z_{l,l'}$ can produce new interesting types of the small-(mr) behaviour, rather than simple power-like one. It is not impossible that the matrix (2.28) for some $\rho_{1,2}$ had a pair of complex conjugate eigenvalues, Δ and Δ^*. Then the scaling function $\xi(mr)$ entering into (2.24) would involve oscillating terms of the form

$$(mr)^{\mathrm{Re}\Delta}\{C_1 \cos[\mathrm{Im}\Delta(mr)] + C_2 \sin[\mathrm{Im}\Delta(mr)]\}, \tag{3.22}$$

with some constants C_i.

Another exotic situation emerges if the matrix $Z_{l,l'}$ cannot be diagonalized and is only reduced to the Jordan form. In this case, the corresponding contribution to the scaling function would involve a logarithmic correction to the power-like behaviour, $(mr)^{\Delta}[C_1 \ln(mr) + C_2]$, where Δ is the eigenvalue related to the Jordan cell. However, these interesting hypothetical possibilities are not actually realized for the special cases studied in [39, 40], in spite of the fact that the structure functions of rather high orders were studied (up to 52 in [40]).

For a given n, the set of eigenvalues of the matrix Δ_F changes continuously with $\rho_{1,2}$ and coincides with the set $\Delta_{n,l}$ from (3.13) at $\rho_{1,2} = 0$. This allows one to label the eigenvalues by the pair of numbers n, l, although for finite $\rho_{1,2}$ they are not determined by the single operator $F_{n,l}$ from (3.12). With these reservations, the analysis of [39, 40] shows that the hierarchy (3.11), obeyed by the dimensions (3.13) at $\rho_{1,2} = 0$, survives for all finite (and not small) values of the anisotropy parameters $\rho_{1,2}$: the leading term for any given n is always determined by the eigenvalue labelled by $l = 0$, the leading correction exponent is labelled by $l = 2$ and so on. What is more, the leading terms appear less sensitive to the anisotropy than the correction ones: for small $\rho_{1,2}$, the correction to the dimension $\Delta_{n,0}$ from (3.13) has the order of $O(\rho^2)$ and not $O(\rho)$ as could be expected. These properties also remain valid for various models (3.18) different from the two-parameter case (3.21).

As a rule, Kraichnan's model is discussed because of the insight it offers into the inertial-range behaviour of the real turbulence. On the other hand, it can be viewed as an interesting example of a nontrivial field-theoretic model, in which some quantities can be calculated nonperturbatively, without the expansion in ε or in the coupling constant. As already mentioned above, the exponents ζ_{2l} from (3.8) for the second-order structure function S_2 were derived exactly within the zero-mode approach for the scalar Kraichnan model [22, 81] and its generalizations to the compressible [38] and vector [42, 48, 49] cases. In the RG and OPE approach, the exponent ζ_{2l} is determined by the scaling dimension of the lth rank tensor operator built of two fields θ and l derivatives, which is unique (up to irrelevant contributions having the form of total derivatives):

$$\theta(x)\partial_{i_1}\cdots\partial_{i_l}\theta(x)+\cdots, \tag{3.23}$$

where the dots stand for subtracted terms with Kronecker delta symbols which make the tensor (3.23) irreducible [84]. Namely, $\zeta_{2l}=l+\gamma_l^*$, where the last term is the anomalous dimension (2.12) of the operator (3.23). The identification was confirmed by the direct one-loop calculation of γ_l^* for the two compressible models (3.1) and (3.2). Therefore, the exact result for ζ_{2l} derived within the zero-mode approach provides a nonperturbative expression (as a function of ε, d and α) for the dimension γ_l^*. This also allows one to derive exact analytic expressions for the dimension $\gamma_l(g)$ away from the fixed point and for the renormalization constant $Z_l(g)$ [84]. Such explicit nonperturbative expressions can be interesting from methodological point of view to discuss validity of perturbation theory, its convergence properties, improved perturbation schemes (e.g. instanton methods [62]) and so on.

4. Velocity ensembles with finite correlation time

Vanishing of the correlation time of the velocity field in Kraichnan's ensemble is crucial for the existence of closed differential equations for the equal-time correlation functions, and hence for the practical applicability of the zero-mode approach. As already mentioned, that approach can be interpreted as an implementation of the famous field-theoretic idea of self-consistent (bootstrap) equations, which involve skeleton diagrams with dressed lines (and probably vertices) and dropped bare terms. Owing to special features of Kraichnan's model (linearity in θ, Gaussianity and time decorrelation of the velocity) such equations can be written explicitly, and they have the forms of differential equations with known coefficient functions written in a closed form. For finite correlation time, their analogues would involve infinite diagrammatic series, so that the corresponding anomalous exponents, to our knowledge, have never been extracted from such equations. A very serious difficulty is that for finite correlation time, such equations necessarily involve different-time correlation functions, which are not Galilean invariant and, therefore, are affected by the infamous 'sweeping effects' that obscure the relevant physical interactions. Systematic elimination of the sweeping effects has always been a notorious problem in the bootstrap approach to the NS turbulence; see e.g. [85]. The first-order corrections in small correlation time to the anomalous exponents were derived for Kraichnan's rapid-change model [63] and analogous shell model [72], but no systematic expansion has been obtained. The transport and dispersion of particles in random Gaussian self-similar velocity fields with finite correlation time were also studied by various analytical or numerical methods in [12–19] and [63–76]; see also the references therein.

Below we confine ourselves with the discussion of the RG and OPE approach to the passive scalar [51–54] and vector [55] advection for a finite-correlated velocity, governed by a synthetic Gaussian velocity ensemble [51–54] and a non-Gaussian velocity described by the stirred NS equation [51, 55].

4.1. Gaussian synthetic velocity ensembles

The most natural generalization of the rapid-change model is the linear stochastic equation for the velocity with given effective **k**-dependent viscosity coefficient and the time-decorrelated random force with given correlator, the model proposed and studied e.g. in [15]. The RG and OPE approach (with no serious alterations in comparison to the vanishing correlation time) is also applicable to such a model [51]. Up to the notation, the energy spectrum of the velocity in the inertial range was taken in the form $E(k) \propto k^{1-2\tilde{\varepsilon}}$, while the correlation time at the wave number k scaled as $t(k) \sim k^{-2+\eta}$. Then $\tilde{\varepsilon}$ and η play the role of two expansion parameters (analogous to the single parameter ε in Kraichnan's case). It was shown that, depending on the values of the exponents $\tilde{\varepsilon}$ and η, the model reveals various types of inertial-range scaling regimes with nontrivial anomalous exponents. For $\eta > \tilde{\varepsilon}$, they coincide with the exponents of the rapid-change model and depend on the only parameter $\varepsilon = 2\tilde{\varepsilon} - \eta$, while for $\tilde{\varepsilon} > \eta$ they coincide with the exponents of the opposite ('quenched' or 'frozen') case and depend only on $\tilde{\varepsilon}$.

The most interesting case is $\eta = \tilde{\varepsilon}$, when the exponents can be nonuniversal through the dependence on the correlation time (more precisely, on the ratio u of the velocity correlation time and the eddy turnover time of the passive scalar). This is in a qualitative agreement with the results of [63, 72] where such nonuniversality was also established. However, due to accidental cancellations, the actual nonuniversality is absent in the one-loop order (studied in [51]) and reveals itself only in the order $O(\varepsilon^2)$, calculated in [54] (including anisotropic sectors).

The main conclusions of [54] can be formulated as follows: the qualitative effect of the finite correlation time on the anomalous scaling depends essentially on the correlation function considered, the value of u and the space dimensionality d. For the low-order structure functions and in low dimensions ($d = 2$ or 3), the inclusion of finite correlation time enhances the intermittency in comparison with both the limits: the time-decorrelated ($u = \infty$) and time-independent ($u = 0$) ones. Although the anomalous exponents have a well-defined limit for $u \to 0$, they show interesting irregularities in the vicinity of the quenched limit: a rapid fall-off when $u = 0$ increases from zero, with infinite slope for $d = 2$, with a pronounced minimum for $u \sim 1$. In contrast, the behaviour in the region of large u is smooth, like for the shell model studied in [72]. For higher order structure functions and large d, the anomalous scaling is always weaker in comparison with the rapid-change limit and the corresponding (positive) correction is maximal for $u = 0$ and monotonically decreases to zero as u tends to infinity.

As was pointed out in [15], the Gaussian model with finite correlation time suffers from the lack of Galilean invariance and therefore misrepresents the self-advection of turbulent eddies. It is well known that the different-time correlations of the Eulerian velocity field are not self-similar, as a result of these 'sweeping effects', and depend substantially on the integral scale. It would be much more appropriate to impose the scaling relations for $\mathcal{E}(k)$ and $t(k)$ in the Lagrangian frame, but this is embarrassing due to the daunting task of relating Eulerian and Lagrangian statistics for a flow with a finite correlation time. In the RG and OPE formalism, the sweeping by the large-scale eddies is related to the contributions of the composite operators built solely of the velocity field $\mathbf{v}(x)$ and its temporal derivatives, as discussed in detail in [8, 30–32] for the case of the stochastic NS equation. In the Gaussian model with the spectrum $E(k) \propto k^{1-2\tilde{\varepsilon}}$, those operators become dangerous (that is, their scaling dimensions become negative) for $\tilde{\varepsilon} \geqslant 1/2$, which gives rise to strong infrared divergences in the correlation functions [51]. This means that the sweeping effects, negligible for small $\tilde{\varepsilon}$s, become important for $\tilde{\varepsilon} \geqslant 1/2$. In a Galilean-invariant model, such operators give no contribution to the quantities like (1.1), as explained in [30–32] for the NS case. In the Gaussian case, these IR divergences persist in the structure functions, which provides not only an upper bound for the reliability of the $\tilde{\varepsilon}$–η expansion, but rather a natural bound for the validity of the Gaussian model itself (which excludes, in particular, the most realistic Kolmogorov's value $\tilde{\varepsilon} = 4/3$ and its vicinity). These conclusions agree with the nonperturbative analysis of [75], where the value of $\tilde{\varepsilon} = 1/2$ was reported as the threshold between two qualitatively different regimes for a Lagrangian particle advected by a Gaussian velocity ensemble. The same threshold value was obtained earlier in [73] for a two-dimensional strongly anisotropic model. In the next sections, we will discuss passive advection by a more realistic velocity ensemble, described by the stirred NS stokes equation; the corresponding model and perturbation theory are manifestly Galilean covariant.

4.2. RG approach to the stochastic Navier–Stokes equation

The RG approach to the stochastic NS equation, pioneered in [86], has a long history. Here we briefly recall the treatment of the problem, based on the standard field-theoretic RG and ε expansion; detailed discussion and more references can be found in [8, 31, 32].

We will discuss the transverse (due to the incompressibility condition $\partial_i v_i = 0$) velocity field satisfying the NS equation with a random driving force

$$\nabla_t v_i = \nu_0 \partial^2 v_i - \partial_i \mathcal{P} + f_i, \tag{4.1}$$

where $\nabla_t = \partial_t + v_i \partial_i$ is the Lagrangian derivative (cf equation (2.1)), and $\mathcal{P}$ and f_i are the pressure and the transverse random force per unit mass (all these quantities depend on $x = \{t, \mathbf{x}\}$). We assume for f a Gaussian distribution with zero mean and correlation function

$$\langle f_i(x) f_j(x')\rangle = \frac{\delta(t-t')}{(2\pi)^d} \int_{k\geqslant m} \mathrm{d}\mathbf{k}\, P_{ij}(\mathbf{k}) d_f(k) \exp[\mathrm{i}\mathbf{k}(\mathbf{x}-\mathbf{x}')], \tag{4.2}$$

where $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2$ is the transverse projector, $d_f(k)$ is some function of $k \equiv |\mathbf{k}|$ and model parameters, and d is the dimension of the $\mathbf{x}$ space. The momentum $m = 1/L$, the reciprocal of the integral scale L related to the velocity, provides IR regularization; its precise form is unessential. For simplicity, we will not distinguish it from the integral scale related to the scalar noise in (2.2).

The standard RG formalism is applicable to the problem (4.1), (4.2) if the correlation function of the random force is chosen in the power form

$$d_f(k) = D_0 k^{4-d-2\varepsilon}, \tag{4.3}$$

where $D_0 > 0$ is the positive amplitude factor and the exponent $0 < \varepsilon \leqslant 2$ plays the role of the RG expansion parameter, analogous to that played by ε in equation (2.3). The most realistic value of the exponent is $\varepsilon = 2$: with an appropriate choice of the amplitude, the function (4.3) for $\varepsilon \to 2$ turns to the delta function, $d_f(k) \propto \delta(\mathbf{k})$, which corresponds to the injection of energy to the system owing to interaction with the largest turbulent eddies; for a more detailed justification see [8, 31, 32]. The results of the RG analysis of the model (4.1)–(4.3) are reliable and internally consistent for small ε, while the possibility of their extrapolation to the real value $\varepsilon = 2$ and thus their relevance for the real fluid turbulence is far from obvious; see e.g. [87] for a recent discussion.

The stochastic problem (4.1)–(4.3) is Galilean invariant for all values of the model parameters, including D_0 and ε; this is equally true for the full problem with additional equations (2.1) and (2.2) for the passively advected scalar field. As a consequence, the corresponding perturbation theory is manifestly Galilean covariant: all the exact relations between the correlation functions imposed by the Galilean symmetry (Ward identities) are satisfied order by order. The renormalization procedure does not violate the Galilean symmetry, so that the improved perturbation expansion, obtained with the aid of RG and OPE, remains covariant. This means, in particular, that the Galilean-invariant quantities, for example, the equal-time structure functions (1.1), are not affected by the sweeping (here, the latter becomes important for $\varepsilon \geqslant 3/2$), in contrast with the models with synthetic Gaussian ensembles.

In a standard fashion, one can construct the field-theoretic action for the stochastic problem (4.1)–(4.3), prove its multiplicative renormalizability, derive the corresponding RG equations, establish existence of an IR-attractive fixed point and systematically calculate various scaling dimensions as series in ε. According to the general theorem [56], the action functional has the form

$$S_v(\mathbf{v}', \mathbf{v}) = v' D_v v'/2 + v'(-\nabla_t + \nu_0 \partial^2) v, \tag{4.4}$$

where D_v is the correlation function (4.2) of the random force f_i and all the integrations over $x = \{t, \mathbf{x}\}$ and summations over the vector indices are understood. The auxiliary vector field v'_i (which appear in the field-theoretic action similarly to θ' in (2.5)) is also transverse, $\partial_i v'_i = 0$, which allows one to omit the pressure term on the right-hand side of equation (4.4), as becomes evident after the integration by parts:

$$\int \mathrm{d}t \int \mathrm{d}\mathbf{x}\, v'_i \partial_i \mathcal{P} = -\int \mathrm{d}t \int \mathrm{d}\mathbf{x}\, \mathcal{P}(\partial_i v'_i) = 0.$$

Of course, this does not mean that the pressure contribution can simply be neglected: the field $\mathbf{v}'$ acts as the transverse projector and selects the transverse part of the expressions to which it is contracted in equation (4.4).

The role of the coupling constant is played by the parameter $g_0 \equiv D_0 \big/ \nu_0^3 \propto \Lambda^{2\varepsilon}$, cf equation (2.4) for Kraichnan's model; Λ is the characteristic UV momentum scale. Canonical dimensions of the field v_i are the same as given in table 1 for Kraichnan's ensemble, while the dimensions of the auxiliary field v'_i are as follows: $d^k_{v'} = d + 1$, $d^\omega_{v'} = -1$, $d_{v'} = d - 1$. Then the standard analysis (see section 2.2) shows that, for all $d > 2$, superficial UV divergences can only be present in the 1-irreducible function $\langle v'v \rangle_{\text{1-ir}}$ and the corresponding counterterm reduces to the form $v'\partial^2 v$. In the special case $d = 2$, a new UV divergence appears in the 1-irreducible function $\langle v'v' \rangle_{\text{1-ir}}$. This case requires special attention, see e.g. [91], and from now on we assume $d > 2$. Then the inclusion of the counterterm is reproduced by the multiplicative renormalization of the action (4.4) with the only independent renormalization constant:

$$S_{vR}(\mathbf{v}', \mathbf{v}) = v' D_v v'/2 + v'(-\nabla_t + \nu Z_1 \partial^2) v, \tag{4.5}$$

which can be reproduced by the multiplicative renormalization of the parameters:

$$g_0 = g\mu^{2\varepsilon} Z_g, \qquad \nu_0 = \nu Z_\nu, \qquad Z_\nu = Z_1, \qquad Z_g = Z_1^{-3}. \tag{4.6}$$

Here g and ν are the renormalized parameters and μ is the reference mass (additional arbitrary parameter of the renormalized theory). The last relation in (4.6) follows from the absence of renormalization of the term with D_v in (4.4). The amplitude D_0 in the term with D_v should be expressed in renormalized parameters using the relations $D_0 = g_0 \nu_0^3 = g\mu^{2\varepsilon}\nu^3$. No renormalization of the fields and the 'mass' $m = 1/L$ is needed. The first-order approximation for Z_ν is well known (see e.g. [8, 30, 31, 32, 86]), the $O(g^2)$ contribution can be found in [87]. The standard derivation (similar to that of sections 2.3 and 2.4) leads to RG equations with an IR-stable fixed point.

From the positivity of the canonical dimensions of the fields v_i, v'_i and (2.28), it follows that, for small ε, critical dimensions of all nontrivial composite operators built of these fields and their derivatives are strictly positive. Thus, the leading contribution in the operator product expansions of the types (2.40) and (2.41) are given by the simplest operator $F = 1$ with $\Delta_F = 0$, they are finite for $mr = 0$; contributions of the nontrivial operators only determine corrections, vanishing for $mr \to 0$. We may conclude that there is no anomalous scaling (in the sense of (1.3)) for the correlation functions of the velocity field in the model (4.1)–(4.3) for small ε.

However, numerical simulations of [88, 89] suggest that, as ε increases, the behaviour of the model (4.1)–(4.3) undergoes a qualitative changeover and the scaling of the velocity structure functions becomes anomalous. In the RG language, this probably means that certain Galilean-invariant operators acquire negative critical dimensions for some finite values of ε, close to the physical value $\varepsilon = 2$. Unfortunately, identification of those operators and calculation of their dimensions on the basis of the model (4.1)–(4.3) lies beyond the scope of the present RG technique: the effect takes place for finite, and not small, values of ε, while the dimensions of the operators are known only in the form of the first terms of the expansions in ε (some dimensions are known exactly, but they all remain positive for $\varepsilon \leqslant 2$). Detailed discussion of the critical dimensions of Galilean-invariant operators can be found in [8, 31, 32] and the original papers cited therein. Hopefully, the problem will be solved with the aid of an alternative perturbation theory. The expansion in $1/d$ seems very promising, but so far it has been constructed only for Kraichnan's model and only to the leading order [22]. Attempts were made to modify the model by introducing N replicas of the velocity field and to construct an expansion in $1/N$ [90], but such modifications are inconsistent with the Galilean symmetry.

4.3. Passive advection by the Navier–Stokes velocity ensemble

It was briefly mentioned in [51] and discussed in detail in [55] that, already for infinitesimal values of ε, when the velocity statistics is not yet intermittent, the scalar field, advected by the NS velocity ensemble (4.1)–(4.3), displays a full-scale anomalous scaling behaviour in the sense of (1.3). The corresponding anomalous exponents can be calculated within an RG and OPE approach, in a systematic perturbation expansion in the parameter ε from (4.3). The practical calculation was accomplished to order ε^2 (two-loop approximation), including anisotropic sectors, in [55].

The action field-theoretic functional that corresponds to the full stochastic problem (2.1), (2.2), (4.1)–(4.3) is

$$S(\Phi) = S_v(\mathbf{v}', \mathbf{v}) + \theta' D_\theta \theta'/2 + \theta'(-\nabla_t + u_0 \nu_0 \partial^2)\theta, \tag{4.7}$$

where $u_0 = \kappa_0/\nu_0$ is the inverse Prandtl number, D_θ is the correlation function (2.2) of the random noise f in (2.1) and S_v is the action (4.4) for the problem (4.1)–(4.3). For general d, the only superficially divergent 1-irreducible Green functions are $\langle v'v\rangle_{1\text{-ir}}$ and $\langle\theta'\theta\rangle_{1\text{-ir}}$, the corresponding counterterms reduce to the forms $v'\partial^2 v$ and $\theta'\partial^2\theta$. Thus, the renormalized action has the form

$$S_R(\Phi) = S_{vR}(\mathbf{v}', \mathbf{v}) + \theta' D_\theta \theta'/2 + \theta'(-\nabla_t + u\nu Z_2 \partial^2)\theta \tag{4.8}$$

with S_{vR} from (4.5) (due to the passivity of θ, the constant $Z_1 = Z_\nu$ in S_{vR} and in relations (4.6) remains the same as for the model (4.4)) and a new renormalization constant Z_2. Relations (4.6) are augmented by

$$u_0 = u Z_u, \qquad Z_u = Z_2 Z_1^{-1} \tag{4.9}$$

with Z_ν from (4.5).

The key observation is that the function $\langle\theta'\theta\rangle_{1\text{-ir}}$ that determines the renormalization constant Z_2 (and hence the β function for the new dimensionless coupling constant $u = \kappa/\nu$) does not involve the correlator D_θ. This means that Z_2, and hence $\beta_u = \widetilde{\mathcal{D}}_\mu u$, are the same as in the model without the random noise f in equation (2.1), where they were calculated to first [92, 93] and second [94] orders; the corresponding RG equations have a unique IR-attractive fixed point in the physical region of the couplings $g > 0$ and $u > 0$. Another consequence, which is crucial for the following, is that the correlation function of the noise can be taken from the very beginning in the large-scale form (2.2), and not as a power function like (4.3). Thus, the canonical dimensions of the fields θ and θ' are the same as in table 1 for Kraichnan's model, and *negative* for θ: $d_\theta = -1$. This means that the composite fields built of the derivatives of θ can be dangerous already for infinitesimal values of ε, and the one-loop calculation of the dimensions Δ_n of the operators (2.32) confirms that they are indeed negative (in fact, they coincide with (1.5) up to a factor $3/2$). Thus, the derivation of the anomalous scaling relations for the structure functions (1.3) using the RG and OPE techniques (see sections 2.5 and 2.6 for Kraichnan's model) is equally applicable to the model (2.1), (2.2), (4.1)–(4.3), and the anomalous exponents are identified with the dimensions Δ_n. In the presence of large-scale anisotropy, dimensions of the tensor operators (3.12) come into play; they show the same hierarchy as for Kraichnan's model, see section 3.2. The calculation of the dimensions Δ_n and Δ_{nl} for the model in question to order $O(\varepsilon^2)$ (two-loop approximation) was accomplished in [55]; the results differ from their analogues in Kraichnan's case.

Two main conclusions that can be drown from the RG analysis of the model (2.1), (2.2), (4.1)–(4.3) are the following.

The critical dimensions of all composite operators (2.32) and (3.12), and therefore the corresponding anomalous exponents are independent of the forcing, specified by the correlator (2.2). In particular, this means that they remain unchanged if the stirring noise in equation (2.1) is replaced by an imposed constant gradient, like e.g. in [15, 16]. The role of the forcing is to maintain the steady state of the system and thus to provide nonvanishing amplitudes for the power-law terms with those universal exponents.

This behaviour is similar to that of a passive scalar, advected by the Gaussian velocity ensemble (2.3) with vanishing correlation time. This observation along with exact resemblance in the RG and OPE picture for both the models suggests that for the passive scalar advected by the Navier–Stokes ensemble the concept of zero modes (and thus of statistical conservation laws) is also applicable, although the corresponding equations are not differential and involve infinite diagrammatic series.

The exponents depend on the parameter ε in the correlator of the stirring force (4.2) in the NS equation and on the dimensionality of the $\mathbf{x}$ space d.

For a Gaussian velocity ensembles with finite correlation time, they also depend on the dimensionless ratio of the correlation times of the scalar and velocity fields; see the discussion in section 4.1. In the case at hand, they could depend, in principle, on the analogous dimensionless parameter $u_0 \equiv \kappa_0/\nu_0$, the (inverse) Prandtl number. After solving the RG equations, this parameter is replaced with the corresponding invariant variable, which has exactly the meaning of the ratio of the scalar and velocity correlation times (for a detailed discussion of this point see [51]). However, the analysis of the RG equation shows that in the IR asymptotic range, this parameter tends to a fixed point, whose coordinate u_* depends on d and ε, but not on the initial value u_0. As a result, all the dimensions like Δ_{nl} appear also independent of u_0. In the RG language, the nonuniversality (that is, the dependence on the ratio u_0 or its analogue) of the exponents in the Gaussian model is a consequence of the infinite degeneracy of the IR-stable fixed point. In the NS model, the fixed point is unique, and the exponents appear universal. One may conclude that the nonuniversality of the anomalous exponents in synthetic ensembles, like those discussed in section 4.1, can be an artefact of their Gaussianity, while for the non-Gaussian velocity ensemble, described by the Galilean-covariant NS equation, and hence for the real passive advection the anomalous exponents are universal, that is, independent of the Prandtl number or the ratio of the scalar and velocity correlation times.

Possibility of extrapolation of the results obtained for the model (2.1), (2.2), (4.1)–(4.3) to the physical value $\varepsilon = 2$, their relevance for the real turbulent advection and comparison with experimental data are also briefly discussed in [55].

5. Passively advected vector fields

Discussion of the anomalous scaling in the previous sections has mostly been concentrated on the passive *scalar* advection. Quoting the author of monograph [1], 'there is considerably more life in the large-scale transport of vector quantities' (p 232). New interesting issues which arise in the passive *vector* advection (in particular, due to the presence of the stretching and pressure-like terms) can be studied analytically for the vector analogues of the Kraichnan's rapid-change model; see [19, 40–45, 47–49, 54, 95–98] and references therein.

In these papers, the case of transverse passive $\theta_i(x) \equiv \theta_i(t, \mathbf{x})$ vector case was studied. Below we mostly confine ourselves with the transverse advecting velocity field $\mathbf{v}(x) \equiv \{v_i(x)\}$ with vanishing correlation time; the cases of a non-transverse [42], strongly anisotropic [40] and a Gaussian field with finite correlation time [54] were also studied.

The advection–diffusion equation for a transverse passive vector field with the most general form of the nonlinear term permitted by the Galilean symmetry has the form [43]

$$\partial_t \theta_i + (v_j \partial_j)\theta_i - \mathcal{A}(\theta_j \partial_j)v_i + \partial_i \mathcal{P} = \kappa_0 \Delta\theta_i + f_i, \tag{5.1}$$

where $\mathcal{A}$ is an arbitrary parameter, $\mathcal{P}(x)$ is the analogue of the pressure in equation (4.1), κ_0 is the diffusivity, Δ is the Laplace operator and $f_i(x)$ is a transverse Gaussian stirring force with a correlator similar to (2.2). The velocity is given by the Gaussian ensemble (2.3).

Owing to the transversality conditions, the pressure can be expressed as the solution of the Poisson equation,

$$\Delta\mathcal{P} = (\mathcal{A} - 1)(\partial_i v_j)(\partial_j \theta_i), \tag{5.2}$$

so that for general $\mathcal{A} \neq 1$ equation (5.1) appears nonlocal in space, similarly to the NS equation (4.1) and in contrast with the scalar case (2.1).

The general model (5.1) contains some special cases interesting on their own:

(i) Kazantzev–Kraichnan kinematic dynamo model ($\mathcal{A} = 1$), when the pressure term vanishes and equation (5.1) becomes local;

(ii) the model of passively advected vector impurity ($\mathcal{A} = 0$), which possesses additional symmetry, $\theta \to \theta + \text{const}$, and has an intrinsic formal resemblance with the stochastic NS equation;

(iii) linearized NS equation with prescribed statistics of the background field ($\mathcal{A} = -1$). From the formal point of view, this case shows no special peculiarities in comparison to the case with general $\mathcal{A} \neq 0$.

In these examples, the vector field can have different physical interpretations: magnetic field, weak perturbation of the prescribed background flow, concentration or density of the impurity particles with an internal structure. Below we briefly discuss them separately, paying attention to their relevance to the RG and OPE approach to the turbulence on the whole.

5.1. Passively advected magnetic fields

For the Kazantzev–Kraichnan kinematic dynamo model ($\mathcal{A} = 1$), the field $\theta_i(x)$ is interpreted as the magnetic field in a conducting fluid with the velocity $v_i(x)$. In the full-scale magnetohydrodynamical (MHD) problem, the velocity is governed by the NS equation (4.1) with the additional term quadratic in θ. It arises from the Lorentz force and describes the effects of the magnetic field on the velocity statistics. The dynamical equation for $\theta_i(x)$ is obtained from the Ohm's law for a moving medium and the Maxwell equations neglecting the displacement current and has the form (5.1) with $\mathcal{A} = 1$; see e.g. [99, 100]. Thus, the Kazantzev–Kraichnan model can be viewed as a simplified version to the full MHD problem, in which the Lorentz force is neglected (no feedback on the velocity field, kinematic approximation) and the NS equation is replaced with the Gaussian ensemble (2.3).

This model was extensively studied in connection with the dynamo effect, generation of the large-scale magnetic field due to the energy of turbulent motion [5, 99, 100], which leads to the instability of the stationary state, at least in the kinematic (linear in θ) approximation. For the ensemble (2.3), this effect takes place only for finite ε ($\varepsilon \geqslant 1$ if $d = 3$ [47]) and lies beyond the scope of the perturbation theory; for a detailed discussion see [47, 95–97] and references therein. It is worth noting that the full-scale MHD problem within the RG formalism was studied in [101, 102], see also discussion and more references in [8, 32].

For smaller ε, the steady state is stable and the issue of anomalous scaling can be addressed. The model has no symmetry with respect to the shift of the field θ, so that the composite operators of the forms θ^n have nontrivial scaling dimensions. As a consequence, the relevant quantities that demonstrate pure scaling behaviour are not the structure functions but the correlation functions of the powers of θ, cf section 3.1 for the scalar density field. This is equally true for the general model (5.1) with the important exception $\mathcal{A} = 0$; see section 5.2. The anomalous scaling in such models occurs already for the pair correlator. The corresponding anomalous exponent for the magnetic case was found exactly, using the zero-mode techniques, in [47]; generalizations to anisotropic sectors were derived in [48, 49].

The RG and OPE approach presented in section 2 for the scalar case is directly extended to the general vector model (5.1) and, in particular, to the special case $\mathcal{A} = 1$. The relations similar to (3.7) can be derived for the equal-time pair correlation functions of the powers of θ; the corresponding anomalous exponents Δ_n are identified with the scaling dimensions of operators θ^n with various arrangements of tensor indices. They were derived only to first order in ε and for general d for the scalars $F_n = (\theta_i\theta_i)^n$ [37] and tensors $F_{n,l} = (\theta_i\theta_i)^p\theta_{i_1}\cdots\theta_{i_l}$ with $n = 2p + l$ [41]; generalizations to the compressible velocity field [42] and the general $\mathcal{A}$ [43] were also obtained. For general d and $\mathcal{A}$, the dimension of the tensor $F_{n,l}$ is given by

$$\Delta_{n,l} = \frac{n\varepsilon}{2} + \frac{\varepsilon\mathcal{A}^2(2n(n-1) - (d+1)(n-l)(d+n+l-2))}{2(d^2 + \mathcal{A}^2 + \mathcal{A}d - 3)} + O(\varepsilon); \quad (5.3)$$

it depends explicitly on $\mathcal{A}$ and for $\mathcal{A} = 1$ coincides, of course, with the result derived earlier for the magnetic case.

The analytical results, obtained within both the zero-mode and RG techniques, show that the general pattern of the anomalous scaling for the magnetic vector model, as well as for the general case with $\mathcal{A} \neq 0$, is essentially the same as for the scalar models. In particular, the anisotropic contributions (activated in the presence of large-scale anisotropy) satisfy hierarchical relations analogous to (3.11) and (3.15); compressibility enhances the intermittency and the penetration of anisotropy towards the smaller scales, etc; cf section 3.2.

5.2. *The $\mathcal{A} = 0$ model: effects of pressure and mixing of operators*

The passive vector model (5.1) with $\mathcal{A} = 0$ was introduced in [44, 98] and further studied in [45, 103–105].

The authors of [98] were motivated by the observation that, for $\mathcal{A} = 0$ (and in fact for all $\mathcal{A} \neq 1$), the dynamical equation (5.1) includes a nonlocal pressure term, so that the corresponding exact equations for the equal-time correlation functions (which can also be derived for the vector case) are not differential (like for the scalar or magnetic cases) but integro-differential. In this respect, they resemble the self-consistency equations for the real NS model; hence the term 'linear pressure model' used in [98]. As a result, substituting self-similar expressions for the zero modes into the equations leads to divergences, in contrast with differential equations for the local models. Nevertheless, the authors of [98] have shown that the anomalous exponents for the second-order structure function S_2 can be derived from those equations and graphically presented the results for $d = 3$ (including anisotropic sectors). The calculation procedure of divergent integrals, employed in [98], involves analytical continuation from the region of convergence, and is therefore close to the concept of analytical regularization (see e.g. [8]).

The results of [98] were verified and augmented in [45]. Starting from the Dyson–Wyld equations and the explicit $SO(d)$ covariant decomposition for the pair correlation function, those authors presented the general recipe of deriving nonperturbative exact equations and explicitly obtained decoupled transcendental equations for the scaling exponents, related to different irreducible representations, in d dimensions. This allows one to give a global description of the behaviour of the full set of solutions in isotropic and anisotropic sectors, and to derive analytical results in all sectors to order $O(\varepsilon)$ and for $\varepsilon = 2$ in d dimensions. This picture was illustrated by a few nonperturbative solutions obtained numerically in two and three dimensions for the isotropic and low-order anisotropic sectors; for some of them, results of [98] were corrected. In contradiction with [98], the zero-mode equations were written in momentum representation. In practical calculation of the resulting integrals, the procedure based on the dimensional regularization prescription was applied and justified.

From the RG viewpoints, the model with $\mathcal{A} = 0$ differs seriously from the general vector case with $\mathcal{A} \neq 0$ (as well as from the scalar model), where the anomalous exponents were identified with scaling dimensions of *individual* composite operators. Here, the anomalous scaling is related with the dimensions of *families* of composite operators, which mix heavily in renormalization. In this respect, the model appears much closer to the nonlinear NS equation, where the inertial-range behaviour of structure functions is believed to be related with the Galilean-invariant (and hence built of the velocity gradients) operators, which mix in renormalization (see the discussion in section 4.2 and [31, 32]). This deep formal analogy was the main motivation for the authors of [44, 45, 103–105] who studied the anomalous scaling of the higher order structure functions in the model with $\mathcal{A} = 0$ within the RG and OPE context.

The RG and OPE approach presented in section 2 for the scalar case can be extended to the model in question; the corresponding RG equation has an IR-attractive fixed point (it becomes negative for $d^2 < 3$, which is related to the instability of the model for such d; as discussed in the next subsection for general $\mathcal{A}$). Owing to the symmetry $\theta \to \theta +$const, which distinguishes the model with $\mathcal{A} = 0$ from the general case, the anomalous scaling reveals itself in the structure functions

$$S_{2n}(\mathbf{r}) \equiv \langle [\theta_r(t, \mathbf{x}) - \theta_r(t, \mathbf{x}')]^{2n} \rangle, \tag{5.4}$$

where $\theta_r \equiv \theta_i r_i / r$ is the component of the passive field along the direction $\mathbf{r} = \mathbf{x} - \mathbf{x}'$, an analogue of the stream-wise component of the turbulent velocity field in real experiments. Like for the scalar case, and in contrast to the vector case with $\mathcal{A} \neq 0$, the anomalous exponents for S_{2n} are determined by the scaling dimensions of the composite operators built of the *gradients* of the passive field and having the form $(\partial\theta\partial\theta)^n$.

For $n = 1$, there are two such operators:

$$F_1 = \partial_i\theta_j\partial_i\theta_j, \qquad F_2 = \partial_i\theta_j\partial_j\theta_i. \tag{5.5}$$

For the transverse field θ, the second operator reduces to a total derivative, $F_2 = \partial_i\partial_j(\theta_j\theta_i)$, and its dimension $\Delta_2 = 2+2\Delta_\theta = \varepsilon$ does not appear on the right-hand side of equation (2.41). Like for the scalar case (see the remark below equation (2.35)), the dimension $\Delta_1 = 0$ is found exactly with the aid of certain Schwinger-type equation [45]. So the second-order function is not anomalous: $S_2 \propto r^{2-\varepsilon}$.

For $n = 2$, there are six independent monomials of the desired form, all of which can be obtained from the fourth rank operator $\Phi^{mnps}_{ijkl} \equiv \partial_i\theta_m\partial_j\theta_n\partial_k\theta_p\partial_l\theta_s$ by various contractions of the tensor indices:

$$F_1 = \Phi^{ijkl}_{jilk}, \qquad F_2 = \Phi^{iikk}_{jjll}, \qquad F_3 = \Phi^{ijkk}_{jill},$$
$$F_4 = \Phi^{iijk}_{jkll}, \qquad F_5 = \Phi^{iijj}_{klkl}, \qquad F_6 = \Phi^{iijk}_{jlkl}. \tag{5.6}$$

At first glance, it seems that one can add another independent monomial, $F_7 = \Phi^{ijkl}_{lijk}$, but in fact it reduces to F_1 up to a total derivative:

$$3F_1 - 6F_7 = \partial_i\big[-6\theta_k\Phi^{spi}_{ksp} + 3\theta_k\Phi^{pis}_{skp} + 2\theta_i\Phi^{kps}_{skp}\big], \tag{5.7}$$

where the notation is analogous to that in (5.6).

Expressions (5.6) and (5.7) illustrate the following new serious problems, which the vector model with $\mathcal{A} = 0$ shares with the nonlinear NS equation, and which distinguish them both from the scalar case.

(i) In contrast to the single operator $F_n = (\partial_i\theta\partial_i\theta)^n$ in (2.32) for the scalar field, now one has a family of operators with a given n. They mix in renormalization, and the corresponding renormalization matrix Z_F is neither diagonal nor triangular. The leading anomalous exponent is given by the minimal eigenvalue of the corresponding matrix Δ_F of critical dimensions in (2.28); the other eigenvalues determine the subleading corrections for $(mr) \to 0$ in representations (2.41).

(ii) The number of relevant operators increases rapidly as n grows; such families should be considered separately for different n. Thus, there is little hope to derive an explicit analytical expression for the anomalous exponent Δ_n as a function of n and d, similar to (1.5) for the scalar case (this appears possible, however, for $d = 2$ [104, 105]).

(iii) Not all the monomials $(\partial\theta\partial\theta)^n$ for a given n appear independent. There are nontrivial linear relations between them, which allow one to represent some monomials as certain linear combinations of the others, probably up to total derivatives. In addition to relations valid for all d, there are special relations for integer (most interesting) dimensions. The number of such relations increases when n grows or (integer) d decreases. Before calculating the matrices Z_F and Δ_F, one has to identify the independent monomials and explicitly eliminate the others; otherwise, meaningless spurious eigenvalues (corresponding to the operators which look nontrivial but in fact vanish) will appear among the full set of eigenvalues. This problem was thoroughly investigated in [104] for $d = 2$ and in [103] for $d = 3$.

For the set (5.6), the matrix Δ_F was derived analytically to order $O(\varepsilon)$ (one-loop approximation) and any d in [44]; see also [45]. Only one eigenvalue, however, can be found analytically for general dimension; the other should be calculated numerically for a fixed d. One of them appears negative: $\Delta_2 = -0.55\varepsilon$ for $d = 3$. This means that the function S_4 shows anomalous scaling. From the Hölder inequalities for the functions S_{2n} it then follows that they are anomalous, so that negative dimensions must be present among the eigenvalues for the sets $(\partial\theta\partial\theta)^n$ for all $n \geqslant 2$. They were calculated (in the one-loop approximation and for $d = 3$) to the order n as high as $n = 9$ in [103]. In order to perform this calculation, the author of [103] had to identify the independent relevant monomials for $d = 3$, which is itself an interesting mathematical problem solved in an elegant way. On the other hand, elimination

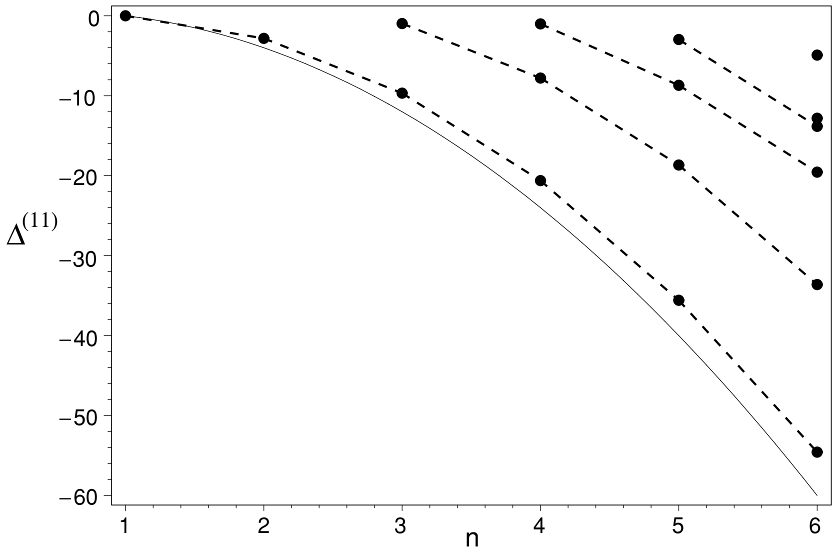

Figure 4. Coefficients $\Delta^{(11)}$ in the $O(\varepsilon/d)$ approximation of the critical dimensions of the operators $(\partial\theta)^{2n}$ in the scalar (solid curve) and vector (thick dots) models. Dashed lines denote monotonous branches of the critical dimensions in the vector case.

of redundant monomials drastically reduces the number of the relevant operators and makes the calculation for high n feasible.

The analysis of the anomalous exponents in the $O(\varepsilon)$ approximation simplifies for large d [45]. To avoid possible confusion, it should be stressed that [45] deals with the $1/d$ expansion of a dimension Δ in its $O(\varepsilon)$ approximation, that is, the $1/d$ expansion of the coefficient $\Delta^{(1)}(d)$ in the representation $\Delta = \varepsilon\Delta^{(1)}(d) + O(\varepsilon^2)$.

It was shown in [45] that, for $d \to \infty$, the negative eigenvalues can be only related to the subset of monomials $(\partial\theta\partial\theta)^n$ in which the vector indices of derivatives are contracted only with each other, and consequently, the indices of fields θ are contracted only with other θs and not with derivatives. In order to find these eigenvalues, it is sufficient to consider the blocks of the full matrices Z_F and Δ_F which correspond to such operators. Of course, this drastically reduces the number of relevant operators in comparison to the general case: there are 2 such operators for $n = 2$, 3 for $n = 3$, 5 for $n = 4$, 7 for $n = 5$, 11 for $n = 6$ [45], 15 for $n = 7$ and 22 for $n = 8$ [106]. The results for the corresponding eigenvalues, derived in [45] to order $n = 6$, confirm and refine the general picture of the anomalous scaling in the vector model: in the full set of operators $(\partial\theta\partial\theta)^k$ with $k \leqslant n$, the most dangerous operator (that is, the operator with the lowest negative dimension) belongs to the subset with $k = n$, and its dimension $\Delta_n < 0$ decreases faster than linearly with n. As functions of n, these leading eigenvalues lie on a well-defined curve, which, however, is not proportional to the factor $n(n-1)$ as it was for the scalar case, cf (1.5). For a given n, the anomalous exponent for the vector model is always 'less negative' than its counterpart for the scalar case. The leading and the next-to-leading correction exponents also lie on well-defined curves, while more distant correction exponents seem to form chaotic patterns, as illustrated by figure 4 taken from [45]. There, the coefficients $\Delta^{(11)}$ in the double expansion for the eigenvalues are shown: $\Delta = \Delta^{(11)}\varepsilon/d+$ corrections.

All these properties, which can be important in the analysis of the stochastic NS problem, become even more pronounced if the results for $n = 7$ and 8 [106] are taken into consideration.

The full analytical treatment of the problem appears possible for $d = 2$, where the transverse vector field can always be represented in the form $\theta_i = \epsilon_{ij}\partial_j\psi$, where ϵ_{ij} is the antisymmetric Levi-Civita pseudotensor and $\psi(x)$ is a scalar function, which again reduces the number of independent operators. In [104], the independent monomials were identified, the matrix Δ_F was reduced to a triangular form and the corresponding eigenvalues were calculated analytically as functions of n to the order $O(\varepsilon)$. The anomalous exponents (minimal eigenvalues) coincide with their counterparts for the scalar model (1.5), but this is an artefact of the one-loop approximation. In [105], it was shown that the matrix Δ_F becomes diagonal to all orders, if the basis operators are not taken as monomials but as powers of only two operators: the local dissipation rate of scalar fluctuations and the enstrophy. The physical meaning of this fact and its relevance for the two-dimensional stochastic NS problem remain to be understood. The authors of [105] performed the $O(\varepsilon^2)$ calculation of the anomalous exponents; the results differ from the scalar case.

5.3. Passive vector model with general $\mathcal{A}$

The passive vector model (5.1) with the velocity ensemble (2.3) was introduced and studied in [43]; generalization to the Gaussian ensemble with compressibility and finite correlation time was studied in [54]. An advantage of the general model is the possibility to control the pressure contribution and thus study its effects on the inertial-range behaviour. It can also be naturally justified within the multiscale technique, as a result of the vertex renormalization [1]. The model exhibits some interesting instabilities, which can be studied nonperturbatively and can be viewed as generalizations of the kinematic dynamo effect in the model with $\mathcal{A} = 1$; see section 5.1.

Like for the case with $\mathcal{A} = 0$, exact transcendental equations can be derived for the exponents ζ_{2l} describing the scaling behaviour of the lth anisotropic 'shell' of second-order structure functions S_2 in the representation; see equations (3.9) and (3.10). Exact solution can only be obtained for the magnetic case $\mathcal{A} = 1$, when the model becomes local and the equations become algebraic. They can be solved numerically or analytically as series in ε or $1/d$. For the exponent of the isotropic shell, one obtains

$$\begin{aligned}
\zeta_{20} = -\mathcal{A}^2\varepsilon + \frac{\mathcal{A}^2\varepsilon}{d}\Bigg\{ & \frac{(\mathcal{A}-1)\Gamma(1+\varepsilon/2)\Gamma(1+(\mathcal{A}^2-1)\varepsilon/2)}{\Gamma(1+\mathcal{A}^2\varepsilon/2)} \\
& - \frac{\varepsilon(\mathcal{A}+1)(\mathcal{A}^3\varepsilon - \mathcal{A}^2\varepsilon + \mathcal{A}\varepsilon + \mathcal{A} - \varepsilon + 1)}{(\mathcal{A}^2\varepsilon - \varepsilon + 2)}\Bigg\} + O(1/d^2) \\
= & -\varepsilon\frac{\mathcal{A}^2(d-1)(d+2)}{(d^2+\mathcal{A}^2+\mathcal{A}d-3)} - \frac{\varepsilon^2\mathcal{A}^2(d-1)}{2d(d^2+\mathcal{A}^2+\mathcal{A}d-3)^2} \\
& \times \{d^3(\mathcal{A}+1)^2 + (d^2-2d+4)(3\mathcal{A}^2+2\mathcal{A}+3)\} + O(\varepsilon^3).
\end{aligned} \tag{5.8}$$

In order ε^3, the function $\psi'(d/2)$ with $\psi(z) \equiv \mathrm{d}\ln\Gamma(z)/\mathrm{d}z$ occurs in the ε expansion; results for general l in the order $O(\varepsilon)$ are given in [43].

The anomalous scaling for the higher order correlation functions is essentially the same as for the magnetic case, see discussion in section 5.1 and expression (5.3) for the anomalous exponents.

Expressions (5.3) and (5.8) show no hint of misbehaviour in the vicinity of the point $\mathcal{A} = 1$, where the pressure term (5.2) vanish and the model (5.1) becomes local; the exponents can regularly be expanded in the parameter $(\mathcal{A}-1)$, which measures the degree of nonlocality. This means that there is no qualitative difference between the local and nonlocal cases.

On the other hand, exponents (5.3) and (5.8) diverge for $(d^2 + \mathcal{A}^2 + \mathcal{A}d - 3) = 0$ (this can happen only for $d \leqslant 2$, and for $d = 2$ only for $\mathcal{A} = -1$). From the RG viewpoints, this is a consequence of the fact that for $(d^2 + \mathcal{A}^2 + \mathcal{A}d - 3) < 0$ the coordinate of the fixed point becomes negative and the solution of the RG equation has no well-defined limit in the IR range. In a more physical language, in that region of parameters the effective diffusivity coefficient is negative at large scales and the system becomes unstable with respect to any small perturbation. For $\mathcal{A} = 0$, this happens if $(d^2 - 3) < 0$, as already mentioned in section 5.2.

Another interesting instability can be revealed by the nonperturbative numerical solution of the exact transcendental equation for the exponent ζ_{20}. There are infinitely many solutions of those equations; some of them are 'admissible' in the terminology of [21, 22] and describe the leading term of the IR behaviour of the pair correlator and the corrections to it, small for $(mr) \to 0$. The other solutions are not admissible and, according to [22], describe the opposite limit $(mr) \to \infty$.

It turns out that the exact real solution for the leading admissible exponent ζ_{20} ceases to exist in some region of the parameters $\mathcal{A}, d$ and ε: it coalesces with the closest inadmissible solution and both become complex [43]. For the magnetic case ($\mathcal{A} = 1$ and $d = 3$), this happens at $\varepsilon = 1$ and is interpreted as the dynamo effect, an instability of the steady state leading to the exponential growth of the pair correlation function [47, 48]. Thus, the analysis of the general model gives the boundaries of the dynamo instability in the extended space of parameters $\mathcal{A}, d$ and ε. In this connection, it is worth noting that coalescence and complexification of nonperturbative exponents of the pair correlation functions takes place for the model with $\mathcal{A} = 0$ [45, 98]. However, in contrast with the situation discussed above, the coalescence occurs only in anisotropic sectors and only for *nonleading admissible* exponents, and one can argue that the steady state remains stable. If this is true, the inertial-range behaviour in the corresponding sectors will include oscillations on the power-like background; cf the discussion of expression (3.22) in section 3.2.

To the best of our knowledge, passive advection of tensor fields has not yet been studied for Kraichnan's ensemble within the RG and OPE approach. From the microscopical point of view, advection–diffusion equations of the forms (2.1), (3.1) and (3.2) describe random walks and transfer of point-like particles. Anomalous scaling for the passive advection of extended objects (polymers or membranes) was studied in [46].

6. Conclusion

Let us briefly summarize the lessons we have learned from the RG and OPE theory of anomalous scaling of passively advected fields.

The zero-mode approach to Kraichnan's rapid-change model gave the first analytic derivation of anomalous scaling based on a microscopical dynamical model. Existence of analytical results (exact solutions, regular perturbation schemes) and accurate numerical simulations allows one to verify and refine classical phenomenological ideas, like the hypothesis on the restored local isotropy of the inertial-range turbulence, and to discuss the issues interesting within the general context of fully developed turbulence: universality and saturation of anomalous exponents, effects of compressibility, anisotropy and pressure, persistence of the large-scale anisotropy and hierarchy of anisotropic contributions, and so on.

Application of the field-theoretic RG and OPE methods gives an alternative derivation of the anomalous scaling and allows one to calculate the anomalous exponents in a systematic perturbation expansions, similar to the ε expansions in the models of critical phenomena. The key difference with the latter is the existence in the corresponding field-theoretic models of infinite number of 'dangerous' composite fields (operators) with negative critical dimensions, which are identified with the anomalous exponents. This gives rise to multiscaling—existence of infinite set of independent anomalous exponents.

Thus, the RG symmetry is not only consistent with multiscaling (in contrast to what was sometimes claimed) but can also be successfully used to establish its existence and to calculate the corresponding infinite set of the anomalous exponents.

Besides the calculational efficiency, an important advantage of the RG approach is its relative universality: it is not restricted to Gaussian velocity ensembles with vanishing correlation time and can also be applied to the case of finite correlation time or non-Gaussian advecting field governed by the stirred NS equation. This is an important step towards the theoretical description of the anomalous scaling for the turbulent velocity field itself.

In spite of the progress achieved in the field of passive advection, systematic derivation of the anomalous exponents on the basis of the stochastic Navier–Stokes equation remains an open question. One can hope that the RG and OPE approach and the concept of dangerous composite fields, combined by an appropriate perturbation theory ($1/d$ expansion?) or some nonperturbative methods (exact renormalization group?, instanton calculus?) will give the satisfactory solution of this interesting problem.

Acknowledgments

I would like to thank the colleagues with whom it has been a pleasure to discuss and to collaborate on the subjects reviewed in this paper: L Ts Adzhemyan, Michal Hnatich, Juha Honkonen, Marian Jurčišin, M V Komarova, M V Kompaniets, Antti Kupiainen, Alessandra Lanotte, Andrea Mazzino, Paolo Muratore Ginanneschi, M Yu Nalimov, S V Novikov, Yu M Pis'mak, A V Runov and A N Vasil'ev. I thank the organizers of the Sixth International Conference 'Renormalization Group 2005' in Helsinki. I also thank the Department of Mathematics in the University of Helsinki for their kind hospitality during my visits, financed by the project 'Extended Dynamical Systems'. The work was supported in part by the Russian Foundation for Fundamental Research (grant no 05-02-17 524).

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